ADDITIVE FUNCTIONS ON $\mathbb{R}$

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0.1. On Homework 6, you were asked to find $S \subset [0,1] \times [0,1]$ with the properties

(i) $\partial S = [0,1] \times [0,1]$, and

(ii) The intersection of S and any vertical or horizontal line consists of at most one point.

Here, we will construct $S \subset \mathbb{R}^2$ with property (ii), but now with (i) replaced by

(i)′ $\partial S = \mathbb{R}^2$.

Before we construct anything, let us make our lives a little easier via the following exercise.

Exercise 0.1.1. Suppose S is a set satisfying property (ii). Then, $S^\circ = \emptyset$. In particular, $\partial S = \bar{S}$, so to verify (i) (resp., (i)′), we need only show that $\bar{S} = [0,1] \times [0,1]$ (resp., $\bar{S} = \mathbb{R}^2$).

If we suggestively reword (ii) to say that S "passes the vertical and horizontal line tests" then we are naturally led to constructing such an S as the graph of a 1-1 function $f : \mathbb{R} \to \mathbb{R}$. First, we will work through a couple warm-up exercises.

Exercise 0.1.2. Show that the pair $(\mathbb{R}, +)$ consisting of the set of real numbers under addition forms a vector space over $\mathbb{Q}$. In particular, $\mathbb{R}$ has a basis as a vector space over $\mathbb{Q}$ (see attached).

Let's pick such a basis, $B = \{x_\alpha\}_{\alpha \in A}$ of $\mathbb{R}$ over $\mathbb{Q}$. For ease, we will suppose that $1 \in B$.

Food for Thought 0.1.3. What's the cardinality of the index set A?

Exercise 0.1.4. Let V and W be two (not necessarily finite dimensional) vector spaces over $\mathbb{Q}$. Pick a basis $B = \{v_\alpha\}_{\alpha \in A}$ of V. Show that given *any* collection of vectors $\{w_\alpha\}_{\alpha \in A} \subset W$, there is a unique linear map $f : V \to W$ such that $f(v_\alpha) = w_\alpha$ for each $\alpha \in A$. (Note that the same indexing set A is used throughout, and that there is nothing special about $\mathbb{Q}$ here.)

Now, we define the function $f : \mathbb{R} \to \mathbb{R}$ whose graph is the desired set S. In fact, we will construct such an f that is linear *as a map of $\mathbb{Q}$-vector spaces*. By Exercise 0.1.4, it suffices to define such a map on the basis vectors x_α.

For starters, let us define $f(1) = 1$.

Exercise 0.1.5. Show that it then follows that $f(r) = r$ for all $r \in \mathbb{Q}$.

So far, the graph of our function will be dense amongst points along the line of slope 1 through the origin. Now, pick your two favorite (distinct) elements $\beta, \gamma \in A$. Define

$$f(x_\alpha) = \begin{cases} x_\alpha & \text{if } \alpha \neq \beta, \\ x_\beta + x_\gamma & \text{if } \alpha = \beta. \end{cases}$$

That's it! We now proceed to prove that the graph, S, of f is indeed dense in $\mathbb{R}^2$. In fact there's nothing special about the function we just constructed. The graph of *any* injective $\mathbb{Q}$-linear map $f : \mathbb{R} \to \mathbb{R}$ that is not $\mathbb{R}$-linear (*i.e.*, not of the form $f(x) = ax$) satisfies properties (i)$'$ and (ii).

Food for Thought 0.1.6. How many such maps are there?

Let $s \in \mathbb{R}$; we think of s as the slope of a line through the origin. We have already shown that points in $\mathbb{R}^2$ on the line of slope $s = 1$ will be in the closure of S. We will show that there is in fact a dense set of slopes with that property. For ease, let us denote by

$$L_s = \{(x, sx) \mid x \in \mathbb{R}\},$$

the graph of the line $y = sx$.

Exercise 0.1.7. Show that for any $0 \neq x \in \mathbb{R}$, the set $\{rx \mid r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.

Exercise 0.1.8. Let $s \in \mathbb{R}$. Show that for any $\epsilon > 0$, there is $s' \in \mathbb{R}$ such that $|s' - s| < \epsilon$, and with the property that $S \cap L_{s'}$ is dense in $L_{s'}$.

Exercise 0.1.9. Conclude that S is dense in $\mathbb{R}^2$.

Exercise 0.1.10. Finally, show that f is one-to-one (in fact, an isomorphism), and conclude that S satisfies the properties (i)$'$ and (ii).

0.2. The function f above is near and dear to my heart, as I spent a long time cooking it up as an undergrad to answer the following open-ended "challenge" problem.

Exercise 0.2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be an *additive* function, that is, a function with the property that $f(a + b) = f(a) + f(b)$ for any $a, b \in \mathbb{R}$. Must f be linear?

Contrast the above exercise with the following.

Exercise 0.2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be both additive and multiplicative (that is, $f(ab) = f(a)f(b)$ for any $a, b \in \mathbb{R}$). Show that f must be linear.