Analysis in $\mathbb{R}^n$
Math 204, Section 30
Winter Quarter 2008
Written Exercises from Week 8

Exercise 0.0.1  (For everyone except Laszlo) If $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is given by
\[ f(x, y, z) = \left( \frac{z}{1+x^2}, \ln(1 + xy^2 z), xye^{-y^2 z}, \sqrt{1000 - x^2 - y^2 - z^2} \right), \]
determine the domain $A \subset \mathbb{R}^3$ on which $f$ is defined, and compute $Df$ for each point of $A$.

Exercise 0.0.2  Prove the Intermediate Value Theorem for Derivatives, which states:
Assume $f : [a, b] \rightarrow \mathbb{R}$ is differentiable at every point (i.e. $f'(c)$ exists for every $c \in (a, b)$ and $f'_+(a)$, the right-hand derivative at $a$, and $f'_-(b)$, the left-hand derivative at $b$, both exist), and that $f'_+(a) \neq f'_-(b)$. If $k$ is any real number between $f'_+(a)$ and $f'_-(b)$, then there is some $x \in (a, b)$ such that $f'(x) = k$.

Exercise 0.0.3  Let $n > 0$, and set $f(x) = x^n \cdot \sin(1/x)$. Determine the values of $m$ for which $f^{(m)}(0)$ exists and for which $f^{(m)}(0)$ is continuous at $x = 0$.

Exercise 0.0.4  For $r \geq 1$, define the function $f_r : \mathbb{R} \rightarrow \mathbb{R}$ as follows:
\[ f_r(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest terms and } x \neq 0 \\ 0, & \text{otherwise} \end{cases} \]
Let $\alpha$ be irrational. Show that $f_r$ is differentiable at $\alpha$ if and only if $r > 2$. 