

EXERCISE 0.3.8...

In the proof of the completeness of l_n^p it is not sufficient to prove just for the case $p = 1$. The reason is that convergence is a topological property but being Cauchy is not. The fact that a p -ball contains a q -ball and vice versa (for $p, q \geq 1$) only tells us that a sequence converges in the p -norm if and only if it converges in the q -norm. It does not guarantee that a sequence is Cauchy in the p -norm will be Cauchy in the q -norm. However, this is true in the case of R^n or C^n , but we need to prove

Lemma 1. *For $p, q \geq 1$, there exists positive constants A and B such that for all $v \in R^n$, we have*

$$A\|v\|_q \leq \|v\|_p \leq B\|v\|_q$$

Proof. Exercise □

This implies that a sequence is Cauchy with respect to the p -norm if and only if it is Cauchy w.r.t the q -norm. Now, it suffices to prove completeness of l_n^p by proving only the case $p = 1$.

To illustrate why we really need the above lemma, here is an example of a space with two metrics that give the same topology but there exist sequences that are Cauchy with respect to one metric but not to the other.

Consider (R, d) , the real number with the usual metric, i.e. $d(x, y) = |x - y|$. Let (R, d') be the real number with a metric d' defined as $d'(x, y) := |f(x) - f(y)|$, where

$$f(x) := \frac{x}{1 + |x|}$$

Show that

1. d' is a metric
2. Any d -ball contains a d' -ball and vice versa. Hence, (R, d) and (R, d') have the same topology.
3. The sequence $x_n = n$ is Cauchy with respect to the d' -metric but clearly not Cauchy in the d -metric. Show that x_n does not converge in (R, d') . Hence, (R, d') is not complete even though it has the same topology as (R, d) .