Analysis in $\mathbb{R}^n$
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Geometric Linear Algebra

Definition 0.0.1 Let $\mathbb{R}$ be the real numbers, that is, the unique ordered field that satisfies the least upper bound property.

Definition 0.0.2 Let $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ be the *points* in $\mathbb{R}^n$. Let $\mathbb{E}^n = \{v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots v_n\mathbf{e}_n \mid v_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$, where $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$, be the *vectors* in $\mathbb{R}^n$.

Definition 0.0.3 If $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{E}^n$, then we define *vector addition* as $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Definition 0.0.4 If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{E}^n$ and $c \in \mathbb{R}$, then we define *scalar multiplication* as $c \cdot \mathbf{x} = (c \cdot x_1, c \cdot x_2, \ldots, c \cdot x_n)$.

Exercise 0.0.5 Show that $\mathbb{E}^n$ is an n-dimensional vector space over $\mathbb{R}$.

Definition 0.0.6 Let $\mathbf{p}_0$ be a point in $\mathbb{R}^n$ and $\mathbf{v}_0$ be a direction in $\mathbb{E}^n$. The *line ℓ through $\mathbf{p}_0$ in the direction $\mathbf{v}_0$* is given by

$$\ell = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} = \mathbf{p}_0 + t\mathbf{v}_0,\ t \in \mathbb{R}\}.$$

Definition 0.0.7 Suppose that $\mathbf{p}_0 \in \mathbb{R}^n$ and $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbb{E}^n$. The *plane through $\mathbf{p}_0$, spanned by $\mathbf{v}$ and $\mathbf{w}$*, is

$$\mathscr{P} = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} = \mathbf{p}_0 + t\mathbf{v} + s\mathbf{w},\ t, s \in \mathbb{R}.\}$$

Definition 0.0.8 If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent vectors in $\mathbb{E}^n$, then we define the *k-dimensional affine subspace through $\mathbf{p}_0 \in \mathbb{R}^n$, spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$*, as

$$\mathbf{H} = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} = \mathbf{p}_0 + t_1\mathbf{v}_1 + \ldots + t_k\mathbf{v}_k,\ \text{where } t_j \in \mathbb{R} \text{ and } 1 \leq j \leq k\}.$$

Definition 0.0.9 When $k = n - 1$ in the definition above, $\mathbf{H}$ is called a *hyperplane* in $\mathbb{R}^n$.

Exercise 0.0.10 If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{E}^n$, then the collection of vectors $\{t_1\mathbf{v}_1 + \ldots + t_k\mathbf{v}_k \mid t_j \in \mathbb{R}, j = 1, \ldots, k\}$ is a subspace of $\mathbb{E}^n$.

Definition 0.0.11 If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent vectors in $\mathbb{E}^n$ with $k \leq n$ and $\mathbf{p}_0 \in \mathbb{R}^n$, we define the *$k$-dimensional parallelepiped with vertex $\mathbf{p}_0$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$* as

$$\mathbf{P} = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} = \mathbf{p}_0 + t_1\mathbf{v}_1 + \ldots + t_k\mathbf{v}_k,\ \text{with } 0 \leq t_j \leq 1\}.$$

Definition 0.0.12 Let V be a vector space over a field F. A *bilinear form* $\langle\cdot,\cdot\rangle$ on V is a map

$$\langle\cdot,\cdot\rangle : V \times V \to F$$

which satisfies linearity in both variables. That is, for all $\mathbf{v}, \mathbf{w} \in V$, and all $\alpha \in F$,

$$\begin{aligned}
\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}\rangle &= \langle \mathbf{v}_1, \mathbf{w}\rangle + \langle \mathbf{v}_2, \mathbf{w}\rangle \\
\langle \alpha\mathbf{v}, \mathbf{w}\rangle &= \alpha\langle \mathbf{v}, \mathbf{w}\rangle \\
\langle \mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2\rangle &= \langle \mathbf{v}, \mathbf{w}_1\rangle + \langle \mathbf{v}, \mathbf{w}_2\rangle \\
\langle \mathbf{v}, \alpha\mathbf{w}\rangle &= \alpha\langle \mathbf{v}, \mathbf{w}\rangle.
\end{aligned}$$

The form $\langle\cdot,\cdot\rangle$ is said to be *symmetric* if $\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$. When $F = \mathbb{R}$ we say the form is *positive definite* if $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ for all $\mathbf{v} \in V$, and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$. Henceforth we assume $F = \mathbb{R}$.

Definition 0.0.13 Suppose that $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are vectors in $\mathbb{E}^n$. The *scalar product* of $\mathbf{v}$ and $\mathbf{w}$ is $\langle \mathbf{v}, \mathbf{w}\rangle = v_1w_1 + \ldots + v_nw_n$. The scalar product is sometimes called the *dot product* and is denoted by $\mathbf{v} \cdot \mathbf{w}$.

Exercise 0.0.14 Prove that the scalar product is a positive definite symmetric bilinear form on $\mathbb{E}^n$.

Definition 0.0.15 If $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{E}^n$, then the *length* or *norm* of $\mathbf{v}$ is defined by

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} = (v_1^2 + \cdots + v_n^2)^{1/2}.$$

Exercise 0.0.16 Prove the following properties of the norm. If $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$, then:

i. $||\mathbf{v}|| \geq 0$;

ii. $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$;

iii. $||\alpha\mathbf{v}|| = |\alpha|\,||\mathbf{v}||, \quad \alpha \in \mathbb{R}$;

iv. $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$;

v. $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right)$.

Theorem 0.0.17 (Cauchy-Schwarz Inequality) Let $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$. Then $|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\|\,\|\mathbf{w}\|$.

Exercise 0.0.18 Prove that equality holds in the Cauchy-Schwarz Inequality iff one of the vectors is a scalar multiple of the other.

Definition 0.0.19 The *distance* between two points $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^n$ is given by $d(\mathbf{p}_1, \mathbf{p}_2) = ||\mathbf{p}_1 - \mathbf{p}_2||$.

Definition 0.0.20 Let $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$. Then $\mathbf{v}$ and $\mathbf{w}$ are said to be *orthogonal* (or *perpendicular*) if $\langle \mathbf{v}, \mathbf{w}\rangle = 0$.

Exercise 0.0.21

i. Show that the $\mathbf{0}$ vector in $\mathbb{E}^n$ is orthogonal to every vector in $\mathbb{E}^n$.

ii. Show that the vectors in the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are pairwise orthogonal, that is $\langle \mathbf{e}_i, \mathbf{e}_j\rangle = 0$ if $i \neq j$, and further that $\langle \mathbf{e}_i, \mathbf{e}_i\rangle = 1$.

iii. If $\mathbf{v}$ is a nonzero vector in $\mathbb{E}^n$, show that the collection $W = \{\mathbf{w} \mid \langle \mathbf{w}, \mathbf{v}\rangle = 0\}$ is an $(n-1)$-dimensional subspace of $\mathbb{E}^n$.

iv. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are pairwise orthogonal non-zero vectors in $\mathbb{E}^n$, show that they form a linearly independent set in $\mathbb{E}^n$.

Theorem 0.0.22 Let $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in $\mathbb{E}^n$. The angle between $\mathbf{v}$ and $\mathbf{w}$ is the unique solution θ to the equation

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{||\mathbf{v}||\,||\mathbf{w}||},\ 0 < \theta < 180^\circ. \tag{1}$$

Definition 0.0.23 Let $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in $\mathbb{E}^n$. The *projection of* $\mathbf{v}$ *onto* $\mathbf{w}$ is defined by

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{||\mathbf{w}||}\frac{\mathbf{w}}{||\mathbf{w}||}. \tag{2}$$

Exercise 0.0.24 Prove that if $\mathbf{w} \neq \mathbf{0}$, then $\mathbf{w}/\|\mathbf{w}\|$ is a unit vector in the direction of $\mathbf{w}$.

Exercise 0.0.25

i. Show that the projection of $\mathbf{v}$ on $\mathbf{w}$ has the same direction as $\mathbf{w}$ if $\langle \mathbf{v}, \mathbf{w}\rangle > 0$, and the direction of $-\mathbf{w}$ if $\langle \mathbf{v}, \mathbf{w}\rangle < 0$.

ii. Show that if $\langle \mathbf{v}, \mathbf{w}\rangle = 0$, then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal and the projection is just the zero vector.

Exercise 0.0.26 Prove that $\|\mathrm{proj}_{\mathbf{w}}(\mathbf{v})\| = \frac{|\langle \mathbf{v}, \mathbf{w}\rangle|}{||\mathbf{w}||}$.

Exercise 0.0.27 Show that $\mathbf{v} - \mathrm{proj}_{\mathbf{w}}(\mathbf{v})$ is orthogonal to $\mathbf{w}$.

Definition 0.0.28 Given two linearly independent vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{E}^3$, we define the *cross-product* to be $\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$.

Exercise 0.0.29

i. Show that $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$.

ii. Show that $\langle \mathbf{v}, \mathbf{v} \times \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v} \times \mathbf{w}\rangle = \mathbf{0}$.

iii. Show that $\mathbf{w} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{w})$.

Exercise 0.0.30

i. Show that $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}||\sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

ii. Show that $\|\mathbf{v} \times \mathbf{w}\|$ is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

Definition 0.0.31 Let F be a subfield of the real numbers, and let $V = F^n$. If $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are elements of V, we say that $\mathbf{v}$ is *orthogonal* (or *perpendicular*) to $\mathbf{w}$ if $\langle \mathbf{v} \mid \mathbf{w}\rangle = \mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n = 0$. We say that $\mathbf{v}$ is a *normalized vector* or *unit vector*, if $\langle \mathbf{v}, \mathbf{v}\rangle = v_1^2 + v_2^2 + \ldots + v_n^2 = 1$. An *orthonormal set* in V is a collection $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of linearly independent vectors in V such that $\langle \mathbf{v}_i \mid \mathbf{v}_j\rangle = 0$ if $i \neq j$, and $\langle \mathbf{v}_i \mid \mathbf{v}_i\rangle = 1$ for all $i, j = 1, \ldots, k$. If $k = n$, the orthonormal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called an *orthonormal basis* for V.

Theorem 0.0.32 Let F be a subfield of $\mathbb{R}$, and let $V = F^n$. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ is a collection of linearly independent vectors in V, where $\mathbf{v}_j = (v_{j1}, v_{j2}, \ldots, v_{jn})$. Consider the matrix

$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ v_{11} & v_{12} & \cdots & v_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{(n-1)1} & v_{(n-1)2} & \cdots & v_{(n-1)n} \end{pmatrix}.$$

Let $\mathbf{v}$ be the vector obtained by taking the determinant of this matrix with respect to the first row. Then $\mathbf{v}$ is nonzero and is orthogonal to each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$. Moreover, the collection $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{v}\}$ is a basis for V.

Exercise 0.0.33 If, as in the previous theorem, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$ forms an orthonormal set in V, show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{v}\}$ is an orthonormal basis for V. (Hint: Consider A^tA and AA^{-1}.)

Exercise 0.0.34 Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ are linearly independent vectors in $\mathbb{E}^n$. Take a point $\mathbf{p}_0$ in $\mathbb{R}^n$ and consider the hyperplane $\mathbf{H}$ through $\mathbf{p}_0$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$. If $\mathbf{v}$ is the vector determined in Theorem 0.0.32, show that $\mathbf{H} = \{\mathbf{p} \in \mathbb{R}^n \mid \langle \mathbf{p} - \mathbf{p}_0, \mathbf{v}\rangle = 0\}$. Specialize this to obtain formulas for a line in $\mathbb{R}^2$ and for a plane in $\mathbb{R}^3$.

Exercise 0.0.35 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ be linearly independent vectors in $\mathbb{E}^n$. Let $\mathbf{p}_0$ be a point of $\mathbb{R}^n$. Let $\mathbf{H}$ be the hyperplane through $\mathbf{p}_0$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$. If $\mathbf{p}$ is any point in $\mathbb{R}^n$, show that the distance from $\mathbf{p}$ to $\mathbf{H}$, that is $\inf\{\|\mathbf{p} - \mathbf{q}\| \mid \mathbf{q} \in \mathbf{H}\}$, is given by the length of the vector $\mathrm{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$ where $\mathbf{v}$ is the vector obtained in Theorem 0.0.32. Specialize this to obtain formulas for the distance from a point to a line in $\mathbb{R}^2$ and from a point to a plane in $\mathbb{R}^3$.

Exercise 0.0.36 Find a formula for the distance from a point to a line in $\mathbb{R}^n$.

Definition 0.0.37 The *Gram-Schmidt Orthogonalization Process* is the following algorithm.

Given a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of linearly independent vectors, we proceed as follows. Let $\tilde{\mathbf{v}}_1 = \mathbf{v}_1$. We continue to find vectors $\tilde{\mathbf{v}}_k$ by taking $\mathbf{v}_k$ and subtracting the projections on the vectors already constructed. More explicitly, we let

$$\begin{aligned}
\tilde{\mathbf{v}}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2). && (3)\\
\tilde{\mathbf{v}}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) - \mathrm{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3). && (4)\\
&\vdots && (5)\\
\tilde{\mathbf{v}}_k &= \mathbf{v}_k - \sum_{i=1}^{k-1} \mathrm{proj}_{\tilde{\mathbf{v}}_i}(\mathbf{v}_k). && (6)
\end{aligned}$$

Exercise 0.0.38 Check that the set of vectors above is pairwise orthogonal, and, in addition, show that $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \ldots, \tilde{\mathbf{v}}_k$ span the same subspace as $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Exercise 0.0.39 Consider the vectors $\mathbf{v}_1 = (1, 1, -1, 0)$ and $\mathbf{v}_2 = (1, 0, 0, -1)$ in $\mathbb{E}^4$, and complete this pair to an orthogonal basis as above.

Exercise 0.0.40 Show that the collection $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{E}^n$.

Exercise 0.0.41 Show that, given linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{E}^n$, we can transform this collection into an orthonormal set $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$, which spans the same subspace. In addition, the set $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_k\}$ can be completed to an orthonormal basis for $\mathbb{E}^n$.