Analysis in $\mathbb{R}^n$
Math 204, Section 30
Winter Quarter 2008

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Metric Spaces

0.1 Compactness, Completeness and Connectedness

Definition 0.1.1 Let A be a nonempty subset of a metric space X. A family $\{U_j\}_{j\in J}$ of open subsets of X is called an *open covering* (or *open cover*) of A if

$$A \subseteq \bigcup_{j\in J} U_j.$$

If $\{U_j\}_{j\in J}$ is an open cover of A, we say that this cover has a *finite subcovering* if there is a finite subcollection $U_{j_1}, U_{j_2}, \ldots, U_{j_n}$ satisfying

$$A \subseteq \bigcup_{k=1}^{n} U_{j_k}.$$

Exercise 0.1.2

i. Let $A = (0,1) \subseteq \mathbb{R}$ with the usual metric. For $j \in \mathbb{N}$, define $U_j = (\frac{1}{j}, 1)$. Then $(0,1) \subseteq \cup_{j\in\mathbb{N}} U_j$, but there is no finite subcover.

ii. Let X be a discrete metric space. For any point $j \in X$, set $U_j = \{j\}$. Then $\{U_j\}_{j\in X}$ is an open cover of X which has a finite subcover iff X is a finite set.

Definition 0.1.3 Let A be a subset of a metric space X. We say that A is *compact* if every open covering of A has a finite subcovering.

For emphasis, we note that the definition insists that for every open covering, there must be a finite subcovering. For example, given any subset A of a metric space X, the set X is an open covering which is already a finite subcovering. So while this particular open covering has a finite subcover, this does not necessarily imply that other open coverings have finite subcoverings.

Theorem 0.1.4 If a subset A of a metric space X is compact, then A is closed and bounded.

Corollary 0.1.5 If A is a compact set in a metric space X, then every infinite subset of A has an accumulation point in A.

Corollary 0.1.6 Let A be a compact set in a metric space. Then, every infinite sequence in A has a subsequence that converges to a point in A.

Exercise 0.1.7

i. Let $f : X \to X'$ be a continuous map of metric spaces. Show that if $A \subseteq X$ is compact, then $f(A) \subseteq X'$ is compact.

ii. Suppose that X is a compact metric space. Show that a continuous function $f : X \to \mathbb{R}$ ($\mathbb{R}$ with the usual metric) is bounded.

iii. Suppose that X is a compact metric space. Show that a continuous function $f : X \to \mathbb{R}$ ($\mathbb{R}$ with the usual metric) attains a maximum and minimum value on X.

Exercise 0.1.8 Suppose X and X' are metric spaces with X compact.

i. If $f : X \to X'$ is continuous on X, show that f is uniformly continuous on X.

ii. If $f : X \to X'$ is a continuous bijection, show that f is a homeomorphism.

Theorem 0.1.9 (Dini's Theorem) Let X be a compact metric space. Suppose f and $(f_n)_{n\in\mathbb{N}}$ are real-valued continuous functions on X. Suppose that, for each $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a monotonic sequence converging to $f(x)$. Show that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly.

Exercise 0.1.10 Suppose that A and B are nonempty subsets of a metric space X. The *distance* between A and B is defined by

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

We say that $d(A, B)$ is *assumed* if there exists $a_0 \in A$ and $b_0 \in B$ such that $d(A, B) = d(a_0, b_0)$. Determine whether or not the distance between A and B is assumed in each of the following cases:

i. A is closed and B is closed;

ii. A is compact and B is closed;

iii. A is compact and B is compact;

iv. What happens in the above cases if we assume X is complete?

Definition 0.1.11 A subset A of a metric space X is *sequentially compact* if every sequence in A has a subsequence that converges to an element of A.

Exercise 0.1.12 If X is a metric space, and $A \subset X$ we say that A is *totally bounded* if, for any $\varepsilon > 0$, A can be covered finite number of balls of radius ε. Show that a sequentially compact metric space is totally bounded.

Lemma 0.1.13 Let X be a metric space. If $A \subset X$ has the property that every infinite subset of A has an accumulation point in X, then there exists a countable collection of open sets $\{U_i \mid i \in \mathbb{N}\}$ such that, if V is any open set in X and $x \in A \cap V$, then there is some U_i such that $x \in U_i \subset V$.

Lemma 0.1.14 Let X be a metric space. If $A \subset X$ has the property that every infinite subset of A has an accumulation point in A, show that for any open covering of A, there exists a countable subcovering.

Theorem 0.1.15 In any metric space, a subset A is compact if and only if it is sequentially compact.

Exercise 0.1.16

i. Show that a compact metric space is complete.

ii. Show that a totally bounded complete metric space is compact.

Theorem 0.1.17 (Heine-Borel) A nonempty subset A of $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the usual metric is compact iff it is closed and bounded.

Exercise 0.1.18 Let B be a compact convex subset of $\mathbb{R}^n$ with the usual metric. Define the *nearest point function* $p : {}^cB \to B$ as follows: For $x \in {}^cB$ we set $p(x)$ to be closest point to x that lies in B. Show that:

i. the function $p(x)$ is well defined;

ii. the point $p(x)$ lies in the boundary of B;

iii. the function $p(x)$ is surjective onto the boundary of B.

In the next exercise, we continue with the terminology of the preceding exercise. Define the supporting hyperplane at $p(x)$ to be the hyperplane through $p(x)$ orthogonal to the vector $p(x)-x$. Define the supporting half space at $p(x)$ to be the set $H_{p(x)} = \{y \in \mathbb{R}^n \mid (y - p(x)) \cdot (p(x) - x) \geq 0\}$.

Exercise 0.1.19

i. Show that, for each $x \in {}^cB$, the set B is a subset of $H_{p(x)}$.

ii. Show that $B = \displaystyle\bigcap_{y \in \partial B} H_y$.

iii. Does the above process work when B is a closed convex unbounded subset of $\mathbb{R}^n$ with the usual metric?

Definition 0.1.20 Let (X, d) be a metric space. A subset $A \subseteq X$ is said to be *dense* in X if $\overline{A} = X$.

Exercise 0.1.21 It is clear that X is dense in X. Is it possible that the only subset of X which is dense in X is X itself?

Definition 0.1.22 Let (X, d) be a metric space. We say that X is *separable* if there exists a countable subset of X which is dense in X.

Exercise 0.1.23 Show that the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with the usual metric are separable.

Theorem 0.1.24 If (X, d) is a compact metric space, then X is separable.

Exercise 0.1.25 Suppose X and X' are metric spaces with X separable. Let $f : X \to X'$ be a continuous surjection. Show that X' is separable.

Definition 0.1.26 Let X be a metric space and let $A \subset X$. We say that A is *not connected* (or *disconnected*)if there exist open sets $U, V \subset X$ such that

a. $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$,

b. $(U \cap A) \cap (V \cap A) = \emptyset$,

c. $A = (U \cap A) \cup (V \cap A)$.

We say that A is *disconnected* by the open sets U and V.

Definition 0.1.27 Let X be a metric space and $A \subset X$. We say A is *connected* if A is not disconnected.

Exercise 0.1.28

i. Show that a subset A of $\mathbb{R}$ in the usual metric is connected iff A is an interval.

ii. Show that a convex subset of $\mathbb{R}^n$ with the usual metric is a connected set.

Theorem 0.1.29 Let X, X' be metric spaces and $f : X \to X'$ a continuous function. If A is a connected subset of X, then $f(A)$ is connected subset of X'. That is, the continuous image of a connected set is connected.

Corollary 0.1.30 (Intermediate Value Theorem) Let X be a metric space, and take $\mathbb{R}$ with the usual metric. Let $f : X \to \mathbb{R}$ be a continuous function. Let A be a connected subset of X and let $I = f(A)$. Then I is an interval in $\mathbb{R}$, and if $x_0 \in I$ there exists $a_0 \in A$ such that $f(a_0) = x_0$.

Exercise 0.1.31 Use the Corollary to show the following. Take $\mathbb{R}$ with the usual metric, and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^n$ for $n \in \mathbb{N}$. If b is a positive real number, show that there exists a unique positive real number a such that $a^n = b$.

Exercise 0.1.32

i. Show that an open ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual metric is a connected set.

ii. Show that the closed ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual metric is a connected set.

iii. Show that $GL(2, \mathbb{R})$ with the metric inherited from $M_2(\mathbb{R})$ is not a connected set. (Hint: use the fact that the determinant is a continuous function.)

iv. Show that $GL(2, \mathbb{C})$ with the metric inherited from $M_2(\mathbb{C})$ is a connected set.

Definition 0.1.33 If X is a metric space and x_0 is in X, then the *connected component of* x_0 *in* X is the union of the connected sets that contain x_0.

Exercise 0.1.34

i. Let X be a metric space and take $x_0 \in X$. Show that the connected component of x_0 is a connected set in X.

ii. Show that if A is a connected subset of X that contains x_0, then A is contained in the connected component of x_0.

iii. Show that if A is a connected subset of a metric space, then $\overline{A}$ is connected.

Exercise 0.1.35 Find the connected components in each of the following metric spaces:

i. $X = \mathbb{R}^\times$, the set of nonzero real numbers with the usual metric.

ii. $X = GL(2, \mathbb{R})$ with the usual metric.

Exercise 0.1.36 Let $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ be metric spaces with the metric inherited from $GL(n, \mathbb{R})$. Show that $O(n, \mathbb{R})$ is not connected and that $SO(n, \mathbb{R})$ is connected.

Definition 0.1.37 A metric space X is *totally disconnected* if the connected component of each point is the point itself.

Example 0.1.38 A discrete metric space X is totally disconnected.

Exercise 0.1.39

i. Find an example of a metric space which is totally disconnected but not discrete.

ii. Find an example of a complete metric space which is totally disconnected but not discrete.