Analysis in $\mathbb{R}^n$
Math 203, Section 30
Autumn Quarter 2007

Paul Sally, e-mail: `sally@math.uchicago.edu`
John Boller, e-mail: `boller@math.uchicago.edu`
website: `http://www.math.uchicago.edu/~boller/M203`

Metric Spaces

0.1 Definition and Basic Properties of Metric Spaces

Definition 0.1.1 A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a map satisfying the following properties.

a. For $x_1, x_2 \in X$, $d(x_1, x_2) \geq 0$;
we have $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$, (positive definite).

b. For any $x_1, x_2 \in X$, we have $d(x_1, x_2) = d(x_2, x_1)$, (symmetric).

c. For any $x_1, x_2, x_3 \in X$, we have

$$d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2),$$

(triangle inequality).

Exercise 0.1.2

i. Draw a triangle and figure out why the triangle inequality is so named.

ii. Replace the triangle inequality by the inequality

$$d(x_1, x_2) \leq d(x_1, x_3) + d(x_2, x_3)$$

for any $x_1, x_2, x_3 \in X$. Show that symmetry follows from this version of the triangle inequality and property a.

Exercise 0.1.3 On $\mathbb{C}^n = \{z = (z_1, z_2, \ldots, z_n) \mid z_j \in \mathbb{C}\}$, we define

$$\|z\| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2}$$

and, for $z, w \in \mathbb{C}^n$, we define $d(z, w) = \|z - w\|$. Show that d is a metric on $\mathbb{C}^n$.

Exercise 0.1.4 Let X be any nonempty set and, for $x_1, x_2 \in X$, define

$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}.$$

Show that d is a metric on X. This is called the *discrete metric*. It is designed to disabuse people of the notion that every metric looks like the usual metric on $\mathbb{R}^n$. The discrete metric is very handy for producing counterexamples.

Definition 0.1.5 We introduce an important collection of metrics on $\mathbb{R}^n$.

Let p be a real number such that $p \geq 1$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}.$$

As usual, if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, we define $d_p(x, y) = \|x - y\|_p$.

Definition 0.1.6 If $I \subset \mathbb{R}$ is an interval, the function $f : I \to \mathbb{R}$ is said to be *convex on I* provided that, given any $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

Lemma 0.1.7 The function $f(x) = e^x$ is convex on $\mathbb{R}$.

Theorem 0.1.8 (Hölder's Inequality) Suppose p, q are real numbers greater than 1 such that $1/p + 1/q = 1$. Suppose $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q \right)^{1/q}$$

(Hint: Let $|x_i| = e^{s/p}$ and $|y_i| = e^{t/q}$.)

Exercise 0.1.9 Now prove that d_p is a metric on $\mathbb{R}^n$.

(Hint: To prove the Triangle Inequality, use Hölder's Inequality and the observation that $(x_i + y_i)^p = x_i(x_i + y_i)^{p-1} + y_i(x_i + y_i)^{p-1}$.)

Exercise 0.1.10 Note that Hölder's inequality works for $p, q > 1$. Prove the triangle inequality for the d_1 metric.

Definition 0.1.11 We also define a metric for $p = \infty$. That is, if $x = (x_1, x_2, \ldots, x_n)$, we set $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$, and define $d_\infty(x, y) = \max_{1 \leq j \leq n} |x_j - y_j| = \|x - y\|_\infty$.

Exercise 0.1.12 Prove that d_∞ defines a metric on $\mathbb{R}^n$.

Definition 0.1.13 The space $(\mathbb{R}^n, d_p)$ or alternatively $(\mathbb{R}^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$, is denoted by $\ell_n^p(\mathbb{R})$. Note that, in our present notation, the norm symbol $\|\cdot\|$ on $\mathbb{R}^n$ should be relabeled $\|\cdot\|_2$.

Exercise 0.1.14 Show that everything we have just done for $\mathbb{R}^n$ can also be done for $\mathbb{C}^n$. This yields a collection of spaces $\ell_n^p(\mathbb{C})$.

0.2 Topology of metric spaces

Definition 0.2.1 Suppose that (X, d) is a metric space and $x_0 \in X$. If $r \in \mathbb{R}$, with $r > 0$, the *open ball of radius r around x_0* is the subset of X defined by $B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}$.

Exercise 0.2.2 In $\mathbb{R}^2$, with the usual metric, illustrate a ball of radius $5/2$ around the point $(-1, 4)$.

Exercise 0.2.3 In $\mathbb{R}^2$, illustrate a ball of radius $5/2$ around the point $(-1, 4)$ in the d_1 metric.

Definition 0.2.4 Suppose that V is a vector space with a metric d. The *unit ball in* V is the ball of radius 1 with center at $\mathbf{0}$, that is $B_1(\mathbf{0})$.

Definition 0.2.5 The *unit ball* in $\ell_n^p(\mathbb{R})$ is the set of all points $x \in \mathbb{R}^n$ such that $\|x\|_p < 1$.

Exercise 0.2.6 For $n = 2$, illustrate the unit balls in $\ell_2^1(\mathbb{R})$, $\ell_2^2(\mathbb{R})$, and $\ell_2^\infty(\mathbb{R})$.

Exercise 0.2.7 If $1 \leq p < q$, show that the unit ball in $\ell_n^p(\mathbb{R})$ is contained in the unit ball in $\ell_n^q(\mathbb{R})$.

Exercise 0.2.8 Consider the set of all points in $\mathbb{R}^2$ which lie outside the unit ball in $\ell_2^1(\mathbb{R})$ and inside the unit ball in $\ell_2^\infty(\mathbb{R})$. Does every point in this region lie on the perimeter of the unit ball in $\ell_2^p(\mathbb{R})$ for some p between 1 and ∞? Do the same problem for $\ell_n^p(\mathbb{R})$.

Definition 0.2.9 Let (X, d) be a metric space and suppose that $A \subseteq X$. The set A is an *open set* in X if, for each $a \in A$, there is an $r > 0$ such that $B_r(a) \subseteq A$.

Exercise 0.2.10 Show that the empty set $\emptyset$ and the whole space X are both open sets.

Exercise 0.2.11 Prove that, for any $x_0 \in X$ and any $r > 0$, the "open ball" $B_r(x_0)$ is open. So now we can legitimately call an "open" ball an open set.

Exercise 0.2.12 Prove that the following are open sets:

i. The "first quadrant," $\{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$, in the usual metric;

ii. any subset of of a discrete metric space.

Theorem 0.2.13

i. If $\{A_j\}_{j \in J}$ is a family of open sets in a metric space (X, d), then

$$\bigcup_{j \in J} A_j$$

is an open set in X;

ii. if $A_1, A_2, \ldots, A_n$ are open sets in a metric space (X, d), then

$$\bigcap_{j=1}^{n} A_j$$

is an open set in X.

Exercise 0.2.14

i. There can be problems with infinite intersections. For example, let $A_n = B_{1/n}((0, 0))$ in $\mathbb{R}^2$ with the usual metric. Show that

$$\bigcap_{n=1}^{\infty} A_n$$

is not open.

ii. Find an infinite collection of distinct open sets in $\mathbb{R}^2$ with the usual metric whose intersection is a nonempty open set.

Definition 0.2.15 Let (X, d) be a metric space and suppose that $A \subseteq X$. We say that A is a *closed set* in X if ${}^cA$ is open in X. (Recall that ${}^cA = X \setminus A$ is the complement of A in X.)

Exercise 0.2.16 Show that the following are closed sets.

i. The x-axis in $\mathbb{R}^2$ with the usual metric;

ii. the whole space X in any metric space;

iii. the empty set in any metric space;

iv. a single point in any metric space;

v. any subset of a discrete metric space.

Exercise 0.2.17 Show that $\mathbb{Q}$ as a subset of $\mathbb{R}$ with the usual metric is neither open nor closed in $\mathbb{R}$. On the other hand, show that if the metric space is simply $\mathbb{Q}$ with the usual metric, then $\mathbb{Q}$ is both open and closed in $\mathbb{Q}$.

Theorem 0.2.18

i. Suppose that (X, d) is a metric space and that $\{A_j\}_{j\in J}$ is a collection of closed sets in X. Then

$$\bigcap_{j\in J} A_j$$

is a closed set in X;

ii. if $A_1, A_2, \ldots, A_n$ are closed sets in X, then

$$\bigcup_{j=1}^{n} A_j$$

is a closed set in X.

Definition 0.2.19 Suppose that A is a subset of a metric space X. A point $x_0 \in X$ is an *accumulation point* of A if, for every $r > 0$, we have $(B_r(x_0) \setminus \{x_0\}) \cap A \neq \emptyset$.

Exercise 0.2.20 Give an example of a metric X and a set $A \subseteq X$ that has at least one accumulation point in A as well as at least one accumulation point not in A.

Definition 0.2.21 Suppose that A is a subset of a metric space X. A point $x_0 \in A$ is an *isolated point* of A if there is an $r > 0$ such that $B_r(x_0) \cap A = \{x_0\}$.

Definition 0.2.22 Suppose that A is a subset of a metric space X. A point $x_0 \in X$ is a *boundary point* of A if, for every $r > 0$, $B_r(x_0) \cap A \neq \emptyset$ and $B_r(x_0) \cap {}^c\!A \neq \emptyset$. The *boundary of* A is the set of boundary points of A, and is denoted by ∂A.

Definition 0.2.23

i. Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$. We take the usual metric on $\mathbb{R}^3$. The set of accumulation points of A is $B^3 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ and is called *the closed unit ball* in $\mathbb{R}^3$ with respect to the usual metric. The set A has no isolated points, and $\partial A = S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. The set S^2 is called the 2-*sphere* in $\mathbb{R}^3$ with respect to the usual metric.

ii. Let $A = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 < 1\}$. We take the usual metric in $\mathbb{R}^n$. The set of accumulation points of A is $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\}$. The set A is called *the open unit ball* with respect to the usual metric and the set B^n is called *the closed unit ball* in $\mathbb{R}^n$ with respect to the usual metric. The set A has no isolated points and $\partial A = S^{n-1} = \{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$. The set S^{n-1} is called the $(n-1)$-*sphere* in $\mathbb{R}^n$ with respect to the usual metric.

Exercise 0.2.24 *i.* Let $A = \mathbb{Q} \subseteq \mathbb{R}$ with the usual metric. Show that every point in $\mathbb{R}$ is an accumulation point of A, the set A has no isolated points, and $\partial A = \mathbb{R}$.

ii. Show that if A is any subset of a discrete metric space X, then A has no accumulation points, that every point in A is an isolated point, and $\partial A = \emptyset$.

Theorem 0.2.25 Suppose A is a subset of a metric space X. Then A is closed iff A contains all its accumulation points.

Exercise 0.2.26 Show that in a discrete metric space any subset is both open and closed.

Exercise 0.2.27 Find an uncountable number of subsets of $\ell_n^p(\mathbb{R})$ and $\ell_n^p(\mathbb{C})$ which are neither open nor closed.

Definition 0.2.28 Suppose that A is a nonempty subset of a metric space X. The *closure of* A is the intersection of all the closed sets which contain A.

Exercise 0.2.29 Show that $A \subseteq \overline{A}$ and $A = \overline{A}$ iff A is closed.

Exercise 0.2.30 For each of the following, find $\overline{A}$:

i. Let $\mathbb{R}^3$ have the usual metric, and let $A = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0\}$.

ii. Let $\mathbb{R}^n$ have the usual metric, and let $A = \mathbb{Q}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_j \in \mathbb{Q} \text{ for } 1 \leq j \leq n\}$.

iii. Let X be a discrete metric space and let A be any subset of X.

Exercise 0.2.31 Suppose that A is a subset of a metric space X. Show that $\overline{A} = A \cup \{\text{accumulation points of } A\}$.

Exercise 0.2.32 Suppose A is a subset of a metric space X. Prove or disprove: $\overline{A} = A \cup \partial A$.

Exercise 0.2.33 Let X be a metric space and let $x_0 \in X$. Suppose that $r > 0$. Prove or disprove: $\overline{B_r(x_0)} = \{x \in X \mid d(x, x_0) \leq r\}$.

Exercise 0.2.34

i. Consider the set of 2×2 matrices over $\mathbb{R}$, that is $M_2(\mathbb{R})$. Make this into a metric space by identifying it with $\mathbb{R}^4$ with the usual metric. Show that $GL_2(\mathbb{R})$ is an open subset of $M_2(\mathbb{R})$ and that $\overline{GL_2(\mathbb{R})} = M_2(\mathbb{R})$.

ii. Show that $SL_2(\mathbb{R})$ is closed subset of $GL_2(\mathbb{R})$

Exercise 0.2.35 Let A be a subset of a metric space X and let x_0 be an isolated point of A. Show that x_0 is in the boundary of A if and only if x_0 is an accumulation point ${}^cA$.

Exercise 0.2.36 As usual, let $\mathbb{R}$ be an ordered field with the least upper bound property. Give $\mathbb{R}$ the discrete metric. Show that $\mathbb{R}$ is still an ordered field with the least upper bound property, but that neither the rational nor irrational numbers are dense in $\mathbb{R}$. Determine what other relevant properties of $\mathbb{R}$ with the usual metric do not hold with the discrete metric.

Definition 0.2.37 Let A be a subset of a metric space X. The *interior* of A is the union of all open sets which are contained in A. We denote the interior of A by A°.

Exercise 0.2.38 Show that $A^\circ \subseteq A$ and $A^\circ = A$ iff A is open.

Exercise 0.2.39

i. Let $X = \mathbb{R}^3$ with the usual metric and $A = \{(x, y, z) \mid z \geq 0\}$. Show that $A^\circ = \{(x, y, z) \mid z > 0\}$;

ii. let X be a discrete metric space and let A be any subset of X. Show that $A^\circ = A$ and $\overline{A} = A$, so that $A = A^\circ = \overline{A}$.

Exercise 0.2.40 Show that, in the usual metric on $\mathbb{R}$, the interior of $\mathbb{Q}$ is empty, that is $\mathbb{Q}^\circ = \emptyset$, but the the interior of $\bar{\mathbb{Q}}$ is $\mathbb{R}$, that is, $(\bar{\mathbb{Q}})^\circ = \mathbb{R}$.

Exercise 0.2.41 Look at combinations of interior, closure, and boundary and determine how many different possibilities result. For this exercise only, let "I" stand for interior, "B" stand for boundary, and "C" stand for closure. Let X be a metric space and let $A \subseteq X$. How many possible sets can be made from A with these operations? For example, $I(I(A)) = I(A)$ but $C(I(A))$ is not necessarily A. Is it $C(A)$? Explore all possibilities of applying combinations of I,C, and B. Hint: There are only a finite number.

Definition 0.2.42 Let A be a nonempty subset of a metric space X. The *diameter of* A is

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y).$$

Exercise 0.2.43

i. Show that the diameter of a set is 0 iff the set consists of a single point.

ii. Suppose A is a nonempty subset of a metric space X. Show that $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.

Definition 0.2.44 Let A be a nonempty subset of $\mathbb{R}^n$. We say that A is *convex* if, given any two points $\mathbf{p}, \mathbf{q} \in A$, the set of points $\{(1-t)\mathbf{p} + t\mathbf{q} \mid t \in \mathbb{R}, 0 \leq t \leq 1\}$ is a subset of A.

Exercise 0.2.45 Show that the unit ball $\ell^p_n(\mathbb{R})$, for $1 \leq p \leq \infty$, is a convex set in $\mathbb{R}^n$.

Definition 0.2.46 Let A be a subset of $\mathbb{R}^n$ with the usual metric. The *convex hull* of A is the intersection of all convex sets containing A. The *closed convex hull* of A is the intersection of all closed convex sets containing A.

Exercise 0.2.47 Let A be a nonempty subset of $\mathbb{R}^n$ and let C be the convex hull of A.

i. Prove or disprove the following statement. The closed convex hull of A is $\overline{C}$.

ii. Show that the diameter of A is the diameter of C.

Exercise 0.2.48

i. Describe of the closed convex hull of the unit ball in $\ell^p_n(\mathbb{R})$ for $1 \leq p \leq \infty$.

ii. Suppose $0 < p < 1$. For $\mathbf{x} \in \mathbb{R}^n$, define,

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$

Define $S_p = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_p \leq 1\}$. Determine whether S_p is convex. If not, find the closed convex hull of S_p.

Definition 0.2.49 Suppose that X is a set and and $F = \mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{B}(X,F)$ the set of all functions from X to F which are bounded. Thus, $f \in \mathcal{B}(X,F)$ iff there is a real number M such that $|f(x)| \leq M$ for all $x \in X$. For $f,g \in \mathcal{B}(X,F)$, we define $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$.

Exercise 0.2.50

i. Let $F = \mathbb{R}$ or $\mathbb{C}$. Show that $\mathcal{B}(X,F)$, with d as defined above, is a metric space.

ii. For $f,g \in \mathcal{B}(X,F)$, define $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. Also, for $\alpha \in F$ define $(\alpha f)(x) = \alpha f(x)$. Show that, with these operations, $\mathcal{B}(X,F)$ is a commutative algebra with 1 over F. Of course, scalar multiplication is simply multiplication by a constant function.

0.3 Limits and Continuous Functions

Definition 0.3.1 Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence of points in a metric space X. We say that a point $L \in X$ is the *limit* of the sequence $(a_n)_{n\in\mathbb{N}}$ as n goes to infinity if, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $d(a_n, L) < \varepsilon$ whenever $n \geq N_\varepsilon$. When the limit exists, we say that $(a_n)_{n\in\mathbb{N}}$ *converges to* L, and write

$$\lim_{n\to\infty} a_n = L.$$

Sometimes, we simply say that $(a_n)_{n\in\mathbb{N}}$ *converges in* X without mentioning L explicitly.

Definition 0.3.2 Let X be a metric space and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(a_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* if, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ whenever $n, m \geq N_\varepsilon$.

Exercise 0.3.3 Suppose that X is a metric space and that the sequence $(a_n)_{n\in\mathbb{N}}$ converges in X. Show that, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ whenever $n, m \geq N_\varepsilon$. Thus, a convergent sequence is a Cauchy sequence.

Exercise 0.3.4 Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a discrete metric space X. Show that there exists $N \in \mathbb{N}$ such that $d(a_n, a_m) = 0$, that is, $a_n = a_m$, for all $n, m \geq N$. Hence, the sequence is convergent. Such a sequence is called *eventually constant.* Note that an eventually constant sequence in any metric space is convergent, and in fact, it converges to the eventual constant.

Definition 0.3.5 Suppose that X is a metric space. We say that X is a *complete metric space* if every Cauchy sequence in X converges.

Examples 0.3.6 The following metric spaces are complete.

i. $\mathbb{R}$ with the usual metric;

ii. $\mathbb{C}$ with the usual metric;

iii. any discrete metric space.

The rational numbers $\mathbb{Q} \subset \mathbb{R}$ are not complete in the usual metric, but they are complete in the discrete metric.

Exercise 0.3.7 Prove that a closed subset of a complete metric space is a complete metric space with the inherited metric.

Exercise 0.3.8 Show that, for $1 \leq p \leq \infty$, the spaces $\ell^p_n(\mathbb{R})$ and $\ell^p_n(\mathbb{C})$ are complete metric spaces.

Lemma 0.3.9 Every bounded sequence in $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the usual metric has a convergent subsequence.

Theorem 0.3.10 (Bolzano-Weierstrass) If A is a bounded infinite subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, then A has an accumulation point.

Definition 0.3.11 Let $\mathcal{B}(X, F)$ denote either $\mathcal{B}(X, \mathbb{R})$ or $\mathcal{B}(X, \mathbb{C})$. There are two types of convergence to be discussed in this space. The first is called *uniform convergence*, that is, convergence with respect to the metric defined above. In this case, a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(X, F)$ is a Cauchy sequence if, given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\sup_{x\in X} |f_n(x) - f_m(x)| < \varepsilon$ for $n, m \geq N_\varepsilon$. On the other hand, for any fixed $x_0 \in X$, set $f(x_0) = \lim_{n\to\infty} f_n(x_0)$. As x_0 varies, this defines a function $f : X \to \mathbb{R}$ or $\mathbb{C}$. This function $f : X \to \mathbb{R}$ or $\mathbb{C}$ is called the *pointwise limit* of the sequence $(f_n)_{n\in\mathbb{N}}$.

Exercise 0.3.12 Show that uniform convergence of a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{B}(X, F)$ implies the existence of a pointwise limit f, or phrased slightly differently, uniform convergence implies pointwise convergence.

Exercise 0.3.13 Define the following sequence of functions in $\mathcal{B}([0,1],\mathbb{R})$.

$$f_n(x) = \begin{cases} 2n^2x & \text{if } 0 \le x \le \frac{1}{2n} \\ -2n^2(x - \frac{1}{n}) & \text{if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}.$$

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function $f(x) = 0$, for every $x \in [0,1]$ but that this convergence is not uniform. (Note that all the functions f_n, as well as the limit function f are continuous by elementary calculus.)

Theorem 0.3.14 The spaces $\mathcal{B}(X,\mathbb{R})$ and $\mathcal{B}(X,\mathbb{C})$ are complete metric spaces.

Definition 0.3.15 Let (X,d) and (X',d') be metric spaces. A function $f : X \to X'$ is *continuous at the point* $x_0 \in X$ if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ whenever $x \in X$ and $d(x,x_0) < \delta$.

Remark 0.3.16 This is the old familiar ε-δ definition. It is simply the statement that

$$\lim_{x\to x_0} f(x) = f(x_0).$$

More generally, we say that

$$\lim_{x\to x_0} f(x) = L$$

for some $L \in X'$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x), L) < \varepsilon$ whenever $0 < d(x,x_0) < \delta$.

Exercise 0.3.17 Suppose that X and X' are metric spaces as above and that $x_0 \in X$. Show that f is continuous at x_0 iff for every sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to x_0 in X, we have

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

in X'.

Remark 0.3.18 Note that another way of saying that f is continuous at x_0 is the following: given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.

Exercise 0.3.19 In discussing continuity, one must be careful about the domain of the function. For example, define $f : \mathbb{R} \to \mathbb{R}$ by the equation

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}.$$

Show that f is not continuous at any point of $\mathbb{R}$. However, suppose we restrict f to be a function from $\mathbb{Q}$ to $\mathbb{Q}$. Show that f is continuous at every point of $\mathbb{Q}$.

Exercise 0.3.20 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ (reduced to lowest terms, } x \neq 0\text{),} \\ 0 & \text{if } x = 0 \text{ or } x \notin \mathbb{Q} \end{cases}.$$

Show that f is continuous at 0 and any irrational point. Show that f is not continuous at any nonzero rational point.

Definition 0.3.21 Continuity is called a *pointwise property* or *local property* of a function f, that is, a function may be continuous at some points, but not at others. We often deal with functions $f : X \to X'$ which are continuous at every point of X. In this case, we simply say that f is *continuous* without reference to any particular point.

Theorem 0.3.22 Suppose that (X, d) and (X', d') are metric spaces. Then a function $f : X \to X'$ is continuous iff for any open set $V \subset X'$, the set $f^{-1}(V)$ is an open set in X.

Exercise 0.3.23

i. Let X and X' be metric spaces and assume that X has the discrete metric. Show that any function $f : X \to X'$ is continuous.

ii. Let $X = \mathbb{R}$ with the usual metric and let $f : X \to X$ be a polynomial function. Show that f is continuous.

iii. Let $X = \mathbb{R}$ with the usual metric and $X' = \mathbb{R}$ with the discrete metric. Describe all continuous functions from $X \to X'$.

Definition 0.3.24 A subset A of a metric space X is *bounded* if there exists a point $x \in X$ and $r > 0$ such that $A \subseteq B_r(x)$.

Exercise 0.3.25 Suppose that (X, d) and (X', d') are metric spaces and that $f : X \to X'$ is continuous. For each of the following statements, determine whether or not it is true. If the assertion is true, prove it. If it is not true, give a counterexample.

i. If A is an open subset of X, then $f(A)$ is an open subset of X';

ii. if B is a closed subset of X', then $f^{-1}(B)$ is a closed subset of X;

iii. if A is a closed subset of X, then $f(A)$ is a closed subset of X';

iv. if A is a bounded subset of X, then $f(A)$ is a bounded subset of X';

v. if B is a bounded subset of X', then $f^{-1}(B)$ is a bounded subset of X;

vi. if $A \subseteq X$ and x_0 is an isolated point of A, then $f(x_0)$ is an isolated point of $f(A)$;

vii. if $A \subseteq X$, $x_0 \in A$, and $f(x_0)$ is an isolated point of $f(A)$, then x_0 is an isolated point of A;

viii. if $A \subseteq X$ and x_0 is an accumulation point of A, then $f(x_0)$ is an accumulation point of $f(A)$;

ix. if $A \subseteq X$, $x_0 \in X$, and $f(x_0)$ is an accumulation point of $f(A)$, then x_0 is an accumulation point of A.

x. Do any of your answers to the above questions change if we assume X and X' are complete?

Definition 0.3.26 Let (X, d) and (X', d') be metric spaces. A continuous function $f : X \to X'$ is a *homeomorphism* if

a. f is a bijection, and

b. the function f^{-1} is also continuous.

Theorem 0.3.27 Suppose $1 \leq p < q \leq \infty$. Then the identity map $I(x) = x$ from $\ell_n^p(\mathbb{R})$ to $\ell_n^q(\mathbb{R})$ is a homeomorphism.

Exercise 0.3.28 Show that $\ell_n^p(\mathbb{C})$ and $\ell_n^q(\mathbb{C})$ are homeomorphic.

Definition 0.3.29 A homeomorphism $f : X \to X'$ is an *isometry* if

$$d'(f(x_1), f(x_2)) = d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Exercise 0.3.30 Suppose that, instead, we had define an isometry to be a bijection $f : X \to X'$ such that $d'(f(x_1), f(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$. Show that with this definition, any isometry is a homeomorphism.

Exercise 0.3.31 Let $X = \mathbb{R}$ with the discrete metric and $X' = \mathbb{R}$ with the usual metric. Define $f : X \to X'$ by $f(x) = x$. Show that f is a continuous bijection which is not a homeomorphism.

Exercise 0.3.32 Let (X, d) be a metric space. Let G be the collection of all homeomorphisms from X to X. Prove that, under composition of functions, G is a group and the collection of all isometries is a subgroup of G.

Exercise 0.3.33 Show that every isometry of $\mathbb{R}$ has the form $f(x) = x + a$ or $f(x) = -x + a$ for some $a \in \mathbb{R}$.

Definition 0.3.34 Suppose that (X, d) is a metric space. Define $\mathcal{BC}(X, F)$ to be the subset of $\mathcal{B}(X, F)$ consisting of continuous functions from X to F. We take the metric on $\mathcal{BC}(X, F)$ to be the same as that on $\mathcal{B}(X, F)$.

Theorem 0.3.35 The space $\mathcal{BC}(X, F)$ is a complete metric space.

Remark 0.3.36 So we have proved that the uniform limit of bounded functions is a bounded function and the uniform limit of bounded continuous functions is a continuous function bounded. We will find these facts very useful in doing analysis.

Exercise 0.3.37 Show that $\mathcal{BC}(X, F)$ is a subalgebra of $\mathcal{B}(X, F)$. That is, $\mathcal{BC}(X, F)$ is a vector subspace of $\mathcal{B}(X, F)$ which is closed under pointwise multiplication.

Exercise 0.3.38 Consider the sequence of functions $f_n : [0, 1] \to [0, 1]$ where $f_n(x) = x^n$. Find the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$ and show that it is not continuous.

Exercise 0.3.39 Define a sequence of functions $f_n : (0, 1) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{q^n} & \text{if } x = \frac{p}{q} \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

for $n \in \mathbb{N}$. Find the pointwise limit f of the sequence $(f_n)_{n\in\mathbb{N}}$ and show that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly.

Definition 0.3.40 Let (X, d) and (X', d') be metric spaces, and let f be a continuous function from X to X'. We say that f is *uniformly continuous* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for any pair $x, y \in X$, we have $d'(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

Remark 0.3.41 Thus, f is uniformly continuous if it is continuous at every point and, for a given $\varepsilon > 0$, we can find a corresponding δ that is independent of the point.

Exercise 0.3.42 Let $X = X' = \mathbb{R}$ with the usual metric.

i. Show that a polynomial function $p(x)$ on $\mathbb{R}$ is uniformly continuous if and only if $\deg(p(x)) < 2$.

ii. Show that $f(x) = \sin(x)$ is uniformly continuous on $\mathbb{R}$.

Exercise 0.3.43 Let $X = (0, \infty)$ and determine whether the following functions are uniformly continuous on X:

i. $f(x) = 1/x$;

ii. $f(x) = \sqrt{x}$;

iii. $f(x) = \ln(x)$;

iv. $f(x) = x \ln(x)$.