Analysis in $\mathbb{R}^n$
Math 205, Section 30
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The Inverse and Implicit Function Theorems

0.1 Inverse and implicit function theorems

Theorem 0.1.1 Suppose that U is an open set in $\mathbb{R}$ and $f : U \to \mathbb{R}$ is C^1. Suppose $x_0 \in U$ is a point such that $f'(x_0) \neq 0$. Show that there is an open interval $I \subset U$ containing x_0 such that:

i. $f : I \to f(I)$ is a bijection;

ii. $J = f(I)$ is an open interval in $\mathbb{R}$; and

iii. $f^{-1} : J \to I$ is differentiable and $(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}$.

Exercise 0.1.2 Find the appropriate domain, range, and derivative of $g(x) = \cos^{-1} x$.

We now state the Inverse Function Theorem, but the lemmas and exercises that follow should be done first.

Theorem 0.1.3 (Inverse Function Theorem) Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be C^1. Let $x_0 \in U$ such that $Df(x_0)$ is nonsingular. Then there exists a neighborhood W of x_0 such that

i. $f : W \to f(W)$ is a bijection;

ii. $f(W)$ is an open set in $\mathbb{R}^n$;

iii. $f^{-1} : f(W) \to W$ is C^1 and $Df^{-1}(f(x)) = (Df(x))^{-1}$ for $x \in W$.

Definition 0.1.4 A *generalized rectangle* in $\mathbb{R}^n$ is a set of the form $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for $i = 1, 2, \ldots, n$.

Lemma 0.1.5 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, and let A° be its interior. Let $f : A \to \mathbb{R}^n$, and let f_i be its i-th coordinate function for each $i = 1, 2, \ldots, n$. Suppose f is of class C^1 and there exists $M > 0$ such that $|D_j f_i(x)| \leq M$ for every $x \in A^\circ$. Then $||f(x) - f(y)|| \leq n^2 M ||x - y||$ for all $x, y \in A$.

(Can you get a better estimate that the constant n^2M?)

Lemma 0.1.6 Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be C^1. Take $x_0 \in U$. Then, given any $\varepsilon > 0$, there exists a neighborhood $V \subset U$ of x_0 such that $|D_j f_i(x) - D_j f_i(x_0)| < \varepsilon$ for every $x \in V$ and every $i, j = 1, 2, \ldots, n$.

Exercise 0.1.7 Show that it is sufficient to prove the Inverse Function Theorem for the case that the linear map $L = Df(x_0)$ is the identity map I by showing that the function $g = L^{-1} \circ f$ satisfies the hypotheses of the theorem if and only if f does, and that $Dg(x_0) = I$.

Lemma 0.1.8 Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be C^1. Take $x_0 \in U$ and suppose that $Df(x_0)$ is nonsingular. Then there exists a neighborhood V of x_0 and a constant $c > 0$ such that

$$||f(y) - f(x)|| \geq c||y - x|| \text{ for all } x, y \in V.$$

(Hint: Consider the function $g(x) = f(x) - x$.)

Exercise 0.1.9 Show that the preceding exercise allows us to assume that f is injective on a closed rectangle $A \subset U$.

Exercise 0.1.10 Show that there exists $d > 0$ such that $||f(x) - f(x_0)|| \geq d$ for every $x \in \partial A$, where A is the closed rectangle from the previous exercise.

Exercise 0.1.11 Let $W = \{y \in \mathbb{R}^n \mid ||y - f(x_0)|| < d/2\}$. Show that for any $y \in W$ and any $x \in \partial A$, we have $||y - f(x_0)|| < ||y - f(x)||$.

Exercise 0.1.12 Show that for any $y \in W$, there is a unique $x \in A^\circ$ such that $f(x) = y$.

(Hint: Minimize $h(x) = ||y - f(x)||^2$ on A, and use the nonsingularity of $Df(x_0)$.)

Exercise 0.1.13 Let $X = A^\circ \cap f^{-1}(W)$. Show that $f : X \to W$ is invertible and that $f^{-1} : W \to X$ is continuous.

Exercise 0.1.14 Show that $f^{-1} : W \to X$ is differentiable.

(Hint #1: For $x \in X$, let $\mu = Df(x)$. Show that f^{-1} is differentiable at $y = f(x)$ with derivative $Df^{-1}(y) = \mu^{-1}$.)

(Hint #2: For $x_1 \in X$, define $\varphi(x_1 - x) = f(x_1) - f(x) - \mu(x_1 - x)$ with μ as in Hint #1.)

Theorem 0.1.15 (Implicit Function Theorem) Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is C^1 on an open set containing (a, b) where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Suppose $f(a, b) = 0$ and that the $m \times m$ matrix $M = (D_{n+j} f_i(a, b))$ is nonsingular. Then that there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b such that, for each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. Furthermore, g is differentiable.

(Hint: Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ be given by $F(x, y) = (x, f(x, y))$ and use the Inverse Function Theorem.)