Analysis in $\mathbb{R}^n$
Math 205, Section 30
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John Boller, e-mail: `boller@math.uchicago.edu`
website: `http://www.math.uchicago.edu/~boller/M203`

Integration

0.1 Riemann Integration in $\mathbb{R}^n$

Definition 0.1.1 A *generalized rectangle* in $\mathbb{R}^n$ is a set of the form $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for $i = 1, 2, \ldots, n$. We also define the *volume* of the rectangle $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ to be $v(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Definition 0.1.2 If $[a, b] \subset \mathbb{R}$ is a closed interval, then a *partition* P of $[a, b]$ is a collection of points $t_0, t_1, \ldots, t_k \in \mathbb{R}$ with $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$.

Definition 0.1.3 If $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is a generalized rectangle, then a *partition* of A is a collection $P = (P_1, P_2, \ldots, P_n)$, where each P_i is a partition of $[a_i, b_i]$. If P_i divides $[a_i, b_i]$ into N_i subintervals, then P divides A into $N_1 \cdot N_2 \cdots N_n$ *subrectangles* in the obvious manner.

Definition 0.1.4 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, let $f : A \to \mathbb{R}$ be a bounded function, and let P be a partition of A. For each subrectangle S of the partition, we define:

$$m_S(f) = \inf\{f(x) \mid x \in S\}$$

$$M_S(f) = \sup\{f(x) \mid x \in S\}$$

Furthermore, we define the *lower sum* and *upper sum* of f corresponding to P as, respectively:

$$L(f, P) = \sum_S m_S(f) \cdot v(S)$$

$$U(f, P) = \sum_S M_S(f) \cdot v(S)$$

Exercise 0.1.5 Show that for any f and P as above, we have $L(f, P) \leq U(f, P)$.

Exercise 0.1.6 Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$. Let $P = (\{0, \frac{1}{2}, 1\}, \{0, \frac{1}{3}, \frac{2}{3}, 1\})$. Compute $L(f, P)$ and $U(f, P)$.

Exercise 0.1.7 Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

For any partition P, compute $L(f, P)$ and $U(f, P)$.

Definition 0.1.8 Let A be a rectangle, and let P a partition of A. We say that the partition P' is a *refinement* of P if every subrectangle of P' is contained in a subrectangle of P.

Lemma 0.1.9 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, let $f : A \to \mathbb{R}$ be a bounded function, and let P be a partition of A. Suppose the partition P' is a refinement of P. Then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

Corollary 0.1.10 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, let $f : A \to \mathbb{R}$ be a bounded function, and let P_1 and P_2 be any two partitions of A. Then $L(f, P_1) \leq U(f, P_2)$.

Definition 0.1.11 If $A \subset \mathbb{R}^n$ is a rectangle, then a bounded function $f : A \to \mathbb{R}$ is called *integrable* on A if

$$\sup_P\{L(f, P)\} = \inf_P\{U(f, P)\}.$$

When f is integrable on A, the number on either side of the above equation is called the *integral* of f on A and is denoted $\int_A f$.

Exercise 0.1.12 Show that a bounded function $f : A \to \mathbb{R}$ is integrable if and only if, given any $\varepsilon > 0$, there is a partition P of A such that $U(f, P) - L(f, P) < \varepsilon$.

Exercise 0.1.13 Decide whether or not the following functions are integrable on the rectangle $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, and determine the integrals of those that are:

i. $f(x, y) = c$, for some constant $c \in \mathbb{R}$

ii. $f(x, y) = \begin{cases} 0 & , \text{ if } x < y \\ 1/2 & , \text{ if } x \geq y \end{cases}$

iii. $f(x, y) = \begin{cases} 0 & , \text{ if } x \text{ is rational} \\ 1 & , \text{ if } x \text{ is irrational} \end{cases}$

iv. $f(x, y) = \begin{cases} 0 & , \text{ if } x \text{ is irrational} \\ 0 & , \text{ if } x \text{ is rational and } y \text{ is irrational} \\ 1/q & , \text{ if } x \text{ is rational and } y = p/q \text{ in lowest terms} \end{cases}$

Exercise 0.1.14 Suppose $f : A \to \mathbb{R}$ is integrable and $g : A \to \mathbb{R}$ differs from f at finitely many points. Show that g is integrable and that $\int_A g = \int_A f$.

Theorem 0.1.15 Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are integrable. Show that $f + g$ is integrable on A and that $\int_A (f + g) = \int_A f + \int_A g$.

Theorem 0.1.16 Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for all $x \in A$. Show that $\int_A f \leq \int_A g$.

Theorem 0.1.17 Suppose $f : A \to \mathbb{R}$ is integrable. Show that $|f|$ is integrable on A and that $|\int_A f| \leq \int_A |f|$.

Definition 0.1.18 A set $X \subset \mathbb{R}^n$ is said to have *measure zero* provided that, given any $\varepsilon > 0$, there exist closed rectangles $\{U_k\}_{k \in \mathbb{N}}$ with $U_k \subset \mathbb{R}^n$ for each k such that $X \subset \bigcup_{k \in \mathbb{N}} U_k$ and $\sum_{k \in \mathbb{N}} v(U_n) < \varepsilon$.

Theorem 0.1.19 Suppose $\{X_i\}_{i \in \mathbb{N}}$ is a countable collection of sets in $\mathbb{R}^n$ that each has measure zero. Then $X = \bigcup_{i \in \mathbb{N}} X_i$ has measure zero.

Definition 0.1.20 A set $X \subset \mathbb{R}^n$ is said to have *Jordan content zero* provided that, given any $\varepsilon > 0$, there exist closed rectangles $\{U_1, \ldots, U_k\}$ with $U_k \subset \mathbb{R}^n$ for each k such that $X \subset \bigcup_{i=1}^{k} U_k$ and $\sum_{i=1}^{k} v(U_n) < \varepsilon$.

Exercise 0.1.21 Show that $[a, b] \subset \mathbb{R}$ does not have Jordan content zero.

Exercise 0.1.22 Show that the set $X = \{(x, 0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ has measure zero but not Jordan content zero.

Theorem 0.1.23 If $X \subset \mathbb{R}^n$ is compact and has measure zero, then X has Jordan content zero.

Exercise 0.1.24 Show that if $X \subset \mathbb{R}^n$ has Jordan content zero, then ∂X has Jordan content zero.

Definition 0.1.25 Let $A \subset \mathbb{R}^n$ be any set, and let $f : A \to \mathbb{R}$ be a bounded function. Let $a \in A$. For any $\delta > 0$, we define:

$$\begin{aligned} M(a, f, \delta) &= \sup\{f(x) \mid x \in A, ||x - a|| < \delta\} \\ m(a, f, \delta) &= \inf\{f(x) \mid x \in A, ||x - a|| < \delta\}. \end{aligned}$$

Then the *oscillation* of f at a is defined to be:

$$o(f, a) = \lim_{\delta \to 0} [M(a, f, \delta) - m(a, f, \delta)].$$

Exercise 0.1.26 Show that the limit defining $o(f, a)$ exists for any point $a \in A$.

Exercise 0.1.27 Find the oscillations of the following functions at the specified points:

i. $f(x) = \begin{cases} x & , \text{ if } x < 0 \\ x + 1 & , \text{ if } x \geq 0 \end{cases}$ at the points $x = 0$ and $x = 3$,

ii. $f(x) = \begin{cases} 1 & , \text{ if } x \in \mathbb{Q} \\ 0 & , \text{ if } x \notin \mathbb{Q} \end{cases}$ at every point $x \in \mathbb{R}$,

iii. $f(x) = \begin{cases} 1/q & , \text{ if } x = p/q \text{ in lowest terms} \\ 0 & , \text{ if } x \notin \mathbb{Q} \end{cases}$ at every point $x \in \mathbb{R}$,

iv. $f(x) = \begin{cases} 0 & , \text{ if } x \leq 0 \\ \sin(\frac{1}{x}) & , \text{ if } x > 0 \end{cases}$ at the point $x = 0$.

Theorem 0.1.28 The bounded function $f : A \to \mathbb{R}$ is continuous at $a \in A$ if and only if $o(f, a) = 0$.

Lemma 0.1.29 Let $A \subset \mathbb{R}^n$ be a closed set. If $f : A \to \mathbb{R}$ is a bounded function and $\varepsilon > 0$, then the set $\{x \in A \mid o(f, x) \geq \varepsilon\}$ is a closed subset of A.

Exercise 0.1.30 Let $A \subset \mathbb{R}^n$ be a closed rectangle. Let $f : A \to \mathbb{R}$ be a bounded function such that $o(f, x) < \varepsilon$ for all $x \in A$. Show that there exists a partition P of A such that $U(f, P) - L(f, P) < \varepsilon \cdot v(A)$.

Theorem 0.1.31 Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be a bounded function. Let $B = \{x \in A \mid f \text{ is not continuous at } x\}$. Then f is integrable on A if and only if B is a set of measure zero. *(Hint: Consider sets of the form* $B_\varepsilon = \{x \in A \mid o(f, x) \geq \varepsilon\}$*.)*

Definition 0.1.32 Let $C \subset \mathbb{R}^n$. The *characteristic function* of the set C is defined by:

$$\chi_C(x) = \begin{cases} 1 & , \text{ if } x \in C \\ 0 & , \text{ if } x \notin C \end{cases}$$

Theorem 0.1.33 If $C \subset \mathbb{R}^n$ is a bounded set and A is a closed rectangle containing C, then $\chi_C : A \to \mathbb{R}$ is integrable on A if and only if ∂C has measure zero.

Exercise 0.1.34 Show that if $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are integrable, then so is $f \cdot g$.

Exercise 0.1.35 Show that every increasing function $f : [a, b] \to \mathbb{R}$ is integrable.