Integration

0.1 Riemann Integration in $\mathbb{R}^n$

Definition 0.1.1 A generalized rectangle in $\mathbb{R}^n$ is a set of the form $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for $i = 1, 2, \ldots, n$. We also define the volume of the rectangle $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ to be $v(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Definition 0.1.2 If $[a, b] \subset \mathbb{R}$ is a closed interval, then a partition $P$ of $[a, b]$ is a collection of points $t_0, t_1, \ldots, t_k \in \mathbb{R}$ with $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$.

Definition 0.1.3 If $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is a generalized rectangle, then a partition of $A$ is a collection $P = (P_1, P_2, \ldots, P_n)$, where each $P_i$ is a partition of $[a_i, b_i]$. If $P_i$ divides $[a_i, b_i]$ into $N_i$ subintervals, then $P$ divides $A$ into $N_1 \cdot N_2 \cdots N_n$ subrectangles in the obvious manner.

Definition 0.1.4 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, let $f : A \to \mathbb{R}$ be a bounded function, and let $P$ be a partition of $A$. For each subrectangle $S$ of the partition, we define:

$$m_S(f) = \inf\{f(x) \mid x \in S\}$$

$$M_S(f) = \sup\{f(x) \mid x \in S\}$$

Furthermore, we define the lower sum and upper sum of $f$ corresponding to $P$ as, respectively:

$$L(f, P) = \sum_S m_S(f) \cdot v(S)$$

$$U(f, P) = \sum_S M_S(f) \cdot v(S)$$

Exercise 0.1.5 Show that for any $f$ and $P$ as above, we have $L(f, P) \leq U(f, P)$.

Exercise 0.1.6 Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$. Let $P = \{\{0, \frac{1}{2}, 1\}, \{0, \frac{1}{3}, \frac{2}{3}, 1\}\}$. Compute $L(f, P)$ and $U(f, P)$.

Exercise 0.1.7 Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \not\in \mathbb{Q} \end{cases}$.

For any partition $P$, compute $L(f, P)$ and $U(f, P)$.
Let $A$ be a rectangle, and let $P$ a partition of $A$. We say that the partition $P'$ is a refinement of $P$ if every subrectangle of $P'$ is contained in a subrectangle of $P$.

Lemma 0.1.9 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, let $f : A \to \mathbb{R}$ be a bounded function, and let $P$ be a partition of $A$. Suppose the partition $P'$ is a refinement of $P$. Then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

Corollary 0.1.10 Let $A \subset \mathbb{R}^n$ be a generalized rectangle, let $f : A \to \mathbb{R}$ be a bounded function, and let $P_1$ and $P_2$ be any two partitions of $A$. Then $L(f, P_1) \leq U(f, P_2)$.

Definition 0.1.11 If $A \subset \mathbb{R}^n$ is a rectangle, then a bounded function $f : A \to \mathbb{R}$ is called integrable on $A$ if

$$\sup_P \{L(f, P)\} = \inf_P \{U(f, P)\}.$$

When $f$ is integrable on $A$, the number on either side of the above equation is called the integral of $f$ on $A$ and is denoted $\int_A f$.

Exercise 0.1.12 Show that a bounded function $f : A \to \mathbb{R}$ is integrable if and only if, given any $\varepsilon > 0$, there is a partition $P$ of $A$ such that $U(f, P) - L(f, P) < \varepsilon$.

Exercise 0.1.13 Decide whether or not the following functions are integrable on the rectangle $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, and determine the integrals of those that are:

i. $f(x, y) = c$, for some constant $c \in \mathbb{R}$

ii. $f(x, y) = \begin{cases} 0 & \text{if } x < y \\ 1/2 & \text{if } x \geq y \end{cases}$

iii. $f(x, y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

iv. $f(x, y) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational and } y \text{ is irrational} \\ 1/q & \text{if } x \text{ is rational and } y = p/q \text{ in lowest terms} \end{cases}$

Exercise 0.1.14 Suppose $f : A \to \mathbb{R}$ is integrable and $g : A \to \mathbb{R}$ differs from $f$ at finitely many points. Show that $g$ is integrable and that $\int_A g = \int_A f$.

Theorem 0.1.15 Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are integrable. Show that $f + g$ is integrable on $A$ and that $\int_A (f + g) = \int_A f + \int_A g$.

Theorem 0.1.16 Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for all $x \in A$. Show that $\int_A f \leq \int_A g$.

Theorem 0.1.17 Suppose $f : A \to \mathbb{R}$ is integrable. Show that $|f|$ is integrable on $A$ and that $|\int_A f| \leq \int_A |f|$.

Definition 0.1.18 A set $X \subset \mathbb{R}^n$ is said to have measure zero provided that, given any $\varepsilon > 0$, there exist closed rectangles $\{U_k\}_{k \in \mathbb{N}}$ with $U_k \subset \mathbb{R}^n$ for each $k$ such that $X \subset \bigcup_{k \in \mathbb{N}} U_k$ and $\sum_{k \in \mathbb{N}} v(U_k) < \varepsilon$.

Theorem 0.1.19 Suppose $\{X_i\}_{i \in \mathbb{N}}$ is a countable collection of sets in $\mathbb{R}^n$ that each has measure zero. Then $X = \bigcup_{i \in \mathbb{N}} X_i$ has measure zero.

Definition 0.1.20 A set $X \subset \mathbb{R}^n$ is said to have Jordan content zero provided that, given any $\varepsilon > 0$, there exist closed rectangles $\{U_1, \ldots, U_k\}$ with $U_k \subset \mathbb{R}^n$ for each $k$ such that $X \subset \bigcup_{i=1}^k U_k$ and $\sum_{i=1}^k v(U_n) < \varepsilon$. 
Exercise 0.1.21  Show that \([a, b] \subset \mathbb{R}\) does not have Jordan content zero.

Exercise 0.1.22  Show that the set \(X = \{(x, 0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2\) has measure zero but not Jordan content zero.

Theorem 0.1.23  If \(X \subset \mathbb{R}^n\) is compact and has measure zero, then \(X\) has Jordan content zero.

Exercise 0.1.24  Show that if \(X \subset \mathbb{R}^n\) has Jordan content zero, then \(\partial X\) has Jordan content zero.

Definition 0.1.25  Let \(A \subset \mathbb{R}^n\) be any set, and let \(f : A \to \mathbb{R}\) be a bounded function. Let \(a \in A\). For any \(\delta > 0\), we define:

\[
M(a, f, \delta) = \sup \{f(x) \mid x \in A, \|x - a\| < \delta\}
\]
\[
m(a, f, \delta) = \inf \{f(x) \mid x \in A, \|x - a\| < \delta\}.
\]

Then the oscillation of \(f\) at \(a\) is defined to be:

\[
o(f, a) = \lim_{\delta \to 0} [M(a, f, \delta) - m(a, f, \delta)].
\]

Exercise 0.1.26  Show that the limit defining \(o(f, a)\) exists for any point \(a \in A\).

Exercise 0.1.27  Find the oscillations of the following functions at the specified points:

i. \(f(x) = \begin{cases} x & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}\) at the points \(x = 0\) and \(x = 3\),

ii. \(f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}\) at every point \(x \in \mathbb{R}\),

iii. \(f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}\) at every point \(x \in \mathbb{R}\),

iv. \(f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sin(\frac{1}{x}) & \text{if } x > 0 \end{cases}\) at the point \(x = 0\).

Theorem 0.1.28  The bounded function \(f : A \to \mathbb{R}\) is continuous at \(a \in A\) if and only if \(o(f, a) = 0\).

Lemma 0.1.29  Let \(A \subset \mathbb{R}^n\) be a closed set. If \(f : A \to \mathbb{R}\) is a bounded function and \(\varepsilon > 0\), then the set \(\{x \in A \mid o(f, x) \geq \varepsilon\}\) is a closed subset of \(A\).

Exercise 0.1.30  Let \(A \subset \mathbb{R}^n\) be a closed rectangle. Let \(f : A \to \mathbb{R}\) be a bounded function such that \(o(f, x) < \varepsilon\) for all \(x \in A\). Show that there exists a partition \(P\) of \(A\) such that \(U(f, P) - L(f, P) < \varepsilon \cdot v(A)\).

Theorem 0.1.31  Let \(A \subset \mathbb{R}^n\) be a closed rectangle, and let \(f : A \rightarrow \mathbb{R}\) be a bounded function. Let \(B = \{x \in A \mid f\) is not continuous at \(x\}\). Then \(f\) is integrable on \(A\) if and only if \(B\) is a set of measure zero. (Hint: Consider sets of the form \(B_\varepsilon = \{x \in A \mid o(f, x) \geq \varepsilon\}\).)

Definition 0.1.32  Let \(C \subset \mathbb{R}^n\). The characteristic function of the set \(C\) is defined by:

\[
\chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}
\]

Theorem 0.1.33  If \(C \subset \mathbb{R}^n\) is a bounded set and \(A\) is a closed rectangle containing \(C\), then \(\chi_C : A \to \mathbb{R}\) is integrable on \(A\) if and only if \(\partial C\) has measure zero.

Exercise 0.1.34  Show that if \(f : A \to \mathbb{R}\) and \(g : A \to \mathbb{R}\) are integrable, then so is \(f \cdot g\).

Exercise 0.1.35  Show that every increasing function \(f : [a, b] \to \mathbb{R}\) is integrable.