Analysis in $\mathbb{R}^n$
Math 205, Section 30
Spring Quarter 2008
John Boller, e-mail: `boller@math.uchicago.edu`
website: `http://www.math.uchicago.edu/∼boller/M203`

Integration

0.1 Fubini's Theorem

Definition 0.1.1 Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be a bounded function. We define the *lower integral* of f to be the supremum of the lower sums of f on A, that is,

$$\mathcal{L}\int_A f = \sup\{L(f, P) \mid P \text{ is a partition of } A\}.$$

SImilarly, we define the *upper integral* of f to be the infimum of the upper sums of f on A, that is,

$$\mathcal{U}\int_A f = \inf\{U(f, P) \mid P \text{ is a partition of } A\}.$$

Exercise 0.1.2 Show that the upper and lower integrals of a bounded function $f : A \to \mathbb{R}$ always exist.

Example 0.1.3 For $f : [0, 1] \to \mathbb{R}$ given by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ is $x \notin \mathbb{Q}$, find $\mathcal{L}\int_A f$ and $\mathcal{U}\int_A f$.

Exercise 0.1.4 Show that a bounded function $f : A \to \mathbb{R}$ is integrable if and only if the upper and lower integrals are equal.

Definition 0.1.5 Let $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^\ell$ be closed rectangles, and let $C = A \times B \subset \mathbb{R}^{k+\ell}$ be the closed rectangle that is their Cartesian product. If $f : C \to \mathbb{R}$ is a bounded function, then we define the iterated integrals of f as follows.

For a fixed $x \in A$, define the function $g_x : B \to \mathbb{R}$ by $g_x(y) = f(x, y)$. Note that g_x may or may not be integrable on B, but when it is, we use the notation $\int_B g_x = \int_B f(x, y)\, dy$ as well as similar expressions for the upper and lower integrals of g_x. Define the auxiliary functions $U : A \to \mathbb{R}$ by $U(x) = \mathcal{U}\int_B g_x$ and $L : A \to \mathbb{R}$ by $L(x) = \mathcal{L}\int_B g_x$.

If U and L are integrable, then the *iterated integrals* of f (with respect to B then A) are given by $\int_A L = \int_A (\mathcal{L}\int_B f(x, y)\, dy)\, dx$ and $\int_A U = \int_A (\mathcal{U}\int_B f(x, y)\, dy)\, dx$.

Theorem 0.1.6 (Fubini's Theorem)

Take the notation from the preceding definitions, if $f : C \to \mathbb{R}$ is integrable, then $L : A \to \mathbb{R}$ and $U : A \to \mathbb{R}$ are integrable, and $\int_C f = \int_A L = \int_A U$.

Example 0.1.7 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 1/2 & , \quad \text{if } x \in \mathbb{Q} \\ y & , \quad \text{if } x \notin \mathbb{Q}. \end{cases}$ With $A = [0,1]$ and $B = [0,1]$, determine whether the iterated integrals $\int_A (\int_B f(x,y)\, dy)\, dx$ and $\int_B (\int_A f(x,y)\, dx)\, dy$ exist. Is f integrable on $A \times B$?

Exercise 0.1.8 Since $f(x,y) = x^2 y$ is continuous, it is integrable on any closed rectangle in $\mathbb{R}^2$. By computing both of its iterated integrals, find $\int_{[2,3]\times[4,5]} f$.

Exercise 0.1.9 Since $f(x,y) = e^{y^2}$ is continuous and the triangle $T = \{(x,y) \in [0,1] \times [0,1] \mid y \geq x\} \subset \mathbb{R}^2$ is a bounded set whose boundary has measure zero, the integral $\int_T f$ exists. Use Fubini's Thoerem to compute the value of this integral.

Exercise 0.1.10

i. Construct an example of a set $A \subset [0,1] \times [0,1]$ such that $\partial A = [0,1] \times [0,1]$ but the intersection of A with each horizontal line and each vertical line in $\mathbb{R}^2$ contains at most one point.

ii. If χ_A is the characteristic function of A, show that $\int_{[0,1]} (\int_{[0,1]} \chi_A(x,y)\, dx)\, dy = 0$ and $\int_{[0,1]} (\int_{[0,1]} \chi_A(x,y)\, dy)\, dx = 0$, but that χ_A is not integrable.