Integration

0.1 Differential Forms

Definition 0.1.1 Let $V$ be a vector space over $\mathbb{R}$. Then $V^k = V \times \cdots \times V = \{(v_1, \ldots, v_k) \mid v_i \in V, 1 \leq i \leq k\}$.

Definition 0.1.2 If $V$ is a vector space over $\mathbb{R}$, a function $T : V^k \to \mathbb{R}$ is called multi-linear (or sometimes $k$-linear) provided that, for each $i = 1, \ldots, k$, we have:

\[
T(v_1, \ldots, v_i + v'_i, \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v'_i, \ldots, v_k)
\]

\[
T(v_1, \ldots, \alpha v_i, \ldots, v_k) = \alpha T(v_1, \ldots, v_i, \ldots, v_k)
\]

for all $v_j \in V$ and all $\alpha \in \mathbb{R}$.

Definition 0.1.3 The space $T^k(V) = \{T : V^k \to \mathbb{R} \mid T \text{ is } k\text{-linear}\}$ is called the collection of $k$-tensors on $V$.

Definition 0.1.4 Let $V$ be a vector space over $\mathbb{R}$. The dual space is the collection of 1-tensors and is denoted $V^* = T^1(V) = \{T : V \to \mathbb{R} \mid T \text{ is linear}\}$.

Exercise 0.1.5

i. Show that $V^*$ is a vector space over $\mathbb{R}$.

ii. If $V$ is finite-dimensional, find $\dim(V^*)$.

iii. If $V$ is finite-dimensional, show that there is a natural isomorphism $V \cong (V^*)^*$.

Exercise 0.1.6 Show that $T^k(V)$ is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined as follows:

\[
(T_1 + T_2)(v_1, \ldots, v_k) = T_1(v_1, \ldots, v_k) + T_2(v_1, \ldots, v_k)
\]

\[
(\alpha T)(v_1, \ldots, v_k) = \alpha T(v_1, \ldots, v_k)
\]

Definition 0.1.7 We define the tensor product of two tensors as follows. If $S \in T^k(V)$ and $T \in T^\ell(V)$, then $S \otimes T \in T^{k+\ell}(V)$, where

\[
S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+\ell}).
\]
Exercise 0.1.8 If $S, S_1, S_2, T, T_1, T_2$ and $U$ are tensors and $\alpha \in \mathbb{R}$, show the following:

i. $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$

ii. $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$

iii. $(\alpha S) \otimes T = \alpha (S \otimes T) = S \otimes (\alpha T)$

iv. $S \otimes (T \otimes U) = (S \otimes T) \otimes U$

Theorem 0.1.9 If $\text{dim}(V) = n$, then the dimension of $T^k(V)$ is $n^k$.

(Hint: Construct tensors in $T^k(V)$ as tensor products of basis elements of the dual space $V^*$.)

Definition 0.1.10 We say that a tensor $T \in T^k(V)$ is symmetric provided that:

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for every $v_1, \ldots, v_k \in V$ and all $1 \leq i < j \leq k$. The collection of symmetric $k$-tensors in $T^k(V)$ is denoted $S^k(V)$.

We say that $T$ is alternating provided that:

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for every $v_1, \ldots, v_k \in V$ and all $1 \leq i < j \leq k$. The collection of alternating $k$-tensors in $T^k(V)$ is denoted $\Lambda^k(V)$.

Exercise 0.1.11 Show that the standard dot product on $V = \mathbb{R}^n$ is a symmetric 2-tensor. Here, we realize the dot product as $T : V \times V \to \mathbb{R}$ given by $T(v_1, v_2) = \langle v_1, v_2 \rangle$.

Exercise 0.1.12 Show that the determinant is an alternating $n$-tensor on $V = \mathbb{R}^n$. Here, we realize the determinant as $D : V^n \to \mathbb{R}$ given by $D(v_1, \ldots, v_n) = \det \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \end{bmatrix}$.

Exercise 0.1.13 Show that $S^2(V) \oplus \Lambda^2(V) = T^2(V)$ by proving the following:

i. $S^2(V)$ and $\Lambda^2(V)$ are subspaces of $T^2(V)$.

ii. $S^2(V) \cap \Lambda^2(V) = \{0\}$

iii. $S^2(V) + \Lambda^2(V) = T^2(V)$

(This means that any $T \in T^2(V)$ may be written as $T = S + A$ with $S \in S^2(V)$ and $A \in \Lambda^2(V)$.)

Definition 0.1.14 If $T \in T^k(V)$, we define the following operators $\text{Sym}$ and $\text{Alt}$ as follows:

$$\text{Sym}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_n} T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

$$\text{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Exercise 0.1.15 Prove the following properties of $\text{Sym}$ and $\text{Alt}$:

i. If $T \in T^k(V)$, then $\text{Sym}(T) \in S^k(V)$.

ii. If $T \in T^k(V)$, then $\text{Alt}(T) \in \Lambda^k(V)$.

iii. If $T \in S^k(V)$, then $\text{Sym}(T) = T$.

iv. If $T \in \Lambda^k(V)$, then $\text{Alt}(T) = T$. 
Exercise 0.1.16  If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, show by example that it is not necessarily the case that $\omega \otimes \eta \in \Lambda^{k+\ell}(V)$.

Definition 0.1.17  If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, we define their wedge product as follows:

$$\omega \wedge \eta = \frac{(k + \ell)!}{k! \ell!} \text{Alt}(\omega \otimes \eta).$$

It is clear from earlier exercises that $\omega \wedge \eta \in \Lambda^{k+\ell}(V)$.

Exercise 0.1.18  If $\omega$ is a $k$-tensor, $\eta$ is an $\ell$-tensor, and $\alpha$ is a scalar, prove the following elementary facts about the wedge product:

i. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$

ii. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$

iii. $(\alpha \omega) \wedge \eta = \alpha (\omega \wedge \eta) = \omega \wedge (\alpha \eta)$

iv. $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$

Theorem 0.1.19

i. If $S \in T^k(V)$ and $T \in T^\ell(V)$ and $\text{Alt}(S) = 0$, then

$$\text{Alt}(S \otimes T) = \text{Alt}(T \otimes S) = 0.$$

ii. If $\omega \in T^k(V)$, $\eta \in T^\ell(V)$, and $\theta \in T^m(V)$, then

$$\text{Alt}(\text{Alt}(\omega \otimes \eta) \otimes \theta) = \text{Alt}(\omega \otimes \eta \otimes \theta) = \text{Alt}(\omega \otimes \text{Alt}(\eta \otimes \theta)).$$

iii. If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$, and $\theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k + \ell + m)!}{k! \ell! m!} \text{Alt}(\omega \otimes \eta \otimes \theta).$$

The implication of part iii. is that the wedge product is associative, and we may simply express any of the three terms as $\omega \wedge \eta \wedge \theta$.

Theorem 0.1.20  Let $V$ be a finite-dimensional vector space with $\text{dim}(V) = n$. If $\{\varphi_1, \ldots, \varphi_n\}$ is a basis for $V^*$, then the set of $k$-tensors $\{\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is a basis for $\Lambda^k(V)$, which thus has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Corollary 0.1.21  $\text{dim}(\Lambda^n(\mathbb{R}^n)) = 1$, and the determinant $D$ (see 0.1.12) is a basis element.

Theorem 0.1.22  Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ with basis $\{v_1, \ldots, v_n\}$, and let $\omega \in \Lambda^n(V)$. If $\{w_1, \ldots, w_n\} \subset V$ with each $w_i = \sum_{j=1}^n a_{ij} v_j$, then

$$\omega(w_1, \ldots, w_n) = det(a_{ij}) \cdot \omega(v_1, \ldots, v_n).$$
Definition 0.1.23  For a fixed $p \in \mathbb{R}^n$, the tangent space to $\mathbb{R}^n$ at $p$ is the set $(\mathbb{R}^n)_p = \{(p, v) \mid v \in \mathbb{R}^n\}$. This is a vector space with addition and scalar multiplication defined by $(p, v_1) + (p, v_2) = (p, v_1 + v_2)$ and $\alpha(p, v) = (p, \alpha v)$, respectively. This vector space has an inner product $\langle \cdot, \cdot \rangle : (\mathbb{R}^n)_p \times (\mathbb{R}^n)_p \to \mathbb{R}$ defined in the natural way by $\langle (p, v), (p, w) \rangle = \langle v, w \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product. Often, for simplicity, we will denote the point $(p, v)$ by $v_p$.

Definition 0.1.24  A vector field on $\mathbb{R}^n$ is a function $F$ that assigns to every $p \in \mathbb{R}^n$ a vector $F(p) \in (\mathbb{R}^n)_p$. If $(e_1)_p, \ldots, (e_n)_p$ is the standard basis in $(\mathbb{R}^n)_p$, then any $F(p)$ may be expressed as $F(p) = F^1(p) \cdot (e_1)_p + \cdots + F^n(p) \cdot (e_n)_p$ for some scalars $F^1(p), \ldots, F^n(p)$. The function $F$ is said to be continuous, differentiable, etc. provided that each of its component functions $F^i$ is.

Definition 0.1.25  A differential $k$-form on $\mathbb{R}^n$ is a function $\omega$ that assigns to every $p \in \mathbb{R}^n$ a $k$-tensor $\omega(p) \in \Lambda^k((\mathbb{R}^n)_p)$. If $(e_1)_p, \ldots, (e_n)_p$ is the standard basis in each $(\mathbb{R}^n)_p$ and $\varphi_1(p), \ldots, \varphi_n(p)$ is the dual basis, then there exist functions $\omega_{i_1, \ldots, i_k} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k}(p) \cdot \varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p).$$

The function $\omega$ is said to be continuous, differentiable, $C^\infty$, etc. provided that each of the functions $\omega_{i_1, \ldots, i_k}$ is.

Remark 0.1.26

1. If $\omega$ and $\eta$ are differential $k$-forms, then we define $\omega + \eta$ in the obvious way.

2. If $\omega$ is a differential $k$-form and $\eta$ is a differential $\ell$-form, then we define $\omega \wedge \eta$ to be a differential $(k + \ell)$-form in the obvious way.

3. If $\omega$ is a differential $k$-form and $f$ is a function, then we define $f \cdot \omega$ in the obvious way. In fact, we may think of $f$ as a differential 0-form, and if we do, then we may denote the product above by $f \cdot \omega = f \wedge \omega$.

Exercise 0.1.27  If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$.
(Don’t write anything down, but think carefully about what $Df(p)$ is.)

Exercise 0.1.28  The map $df$ defined by $df(v_p) = Df(p)(v)$ is a differential 1-form.

Definition 0.1.29  Let $x_i : \mathbb{R}^n \to \mathbb{R}$ denote the $i$-th coordinate projection. That is, if $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then $x_i(v) = v_i$.

Exercise 0.1.30  Show that $\{dx_1(p), \ldots, dx_n(p)\}$ is the dual basis to $\{(e_1)_p, \ldots, (e_n)_p\}$.

Remark 0.1.31  The previous exercise implies that every differential $k$-form $\omega$ may be written as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for some functions $\omega_{i_1, \ldots, i_k}$.

Theorem 0.1.32  If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then $df = D_1 f \cdot dx_1 + \cdots + D_n f \cdot dx_n$. 

Definition 0.1.33 Suppose $f : V \to W$ is a linear map between finite-dimensional real vector spaces. Then, for any $k \in \mathbb{N}$, we have an induced map $f^* : T^k(W) \to T^k(V)$ defined by

$$f^*(T)(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k)),$$

for any $T \in T^k(W)$.

Exercise 0.1.34 Assume $f : V \to W$ is a linear map of finite-dimensional real vector spaces.

i. If $S$ and $T$ are tensors, show that $f^*(S \otimes T) = f^*S \otimes f^*T$.

ii. If $\omega$ and $\eta$ are alternating tensors, show that $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Definition 0.1.35 Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable. Then, for each $p \in \mathbb{R}^n$, we have a linear transformation $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$. This yields a map $f_* : (\mathbb{R}^n)_p \to (\mathbb{R}^m)_{f(p)}$ defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)}.$$

As in Definition 0.1.33 above, this linear map induces a linear map $f^* : \Lambda^k((\mathbb{R}^m)_{f(p)}) \to \Lambda^k((\mathbb{R}^n)_p)$ as follows. If $\omega$ is a differential $k$-form on $\mathbb{R}^m$, then $f^*\omega$ is a differential $k$-form on $\mathbb{R}^n$ given by

$$(f^*\omega)(p) = f^*(\omega(f(p))).$$

In other words, if $v_1, \ldots, v_k \in (\mathbb{R}^n)_p$, then

$$f^*(\omega(p))(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k)).$$

Theorem 0.1.36 Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Then:

i. $f^*(dx_i) = \sum_{j=1}^n D_j f_i \cdot dx_j$

ii. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

iii. $f^*(g \cdot \omega) = (g \circ f) \cdot f^*(\omega)$

iv. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

Theorem 0.1.37 If $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, then

$$f^*(g \cdot dx_1 \wedge \cdots \wedge dx_n) = (g \circ f)(\det Df)dx_1 \wedge \cdots \wedge dx_n.$$

Definition 0.1.38 We define an operator $d$ on differential forms as follows. If $\omega$ is a differential $k$-form defined by

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

then $d\omega$ is a differential $(k + 1)$-form, called the differential of $\omega$, defined by

$$d\omega = \sum_{i_1 < \cdots < i_k} d\omega_{i_1, \ldots, i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^n D_j(\omega_{i_1, \ldots, i_k}) \cdot dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Theorem 0.1.39 Let $\omega$ and $\theta$ be differentiable $k$-forms, $\eta$ a differentiable $\ell$-form, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a differentiable function. Then:

i. $d(\omega + \theta) = d\omega + d\theta$

ii. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \eta \wedge d\omega$

iii. $d(d\omega) = 0$

iv. $f^*(d\omega) = d(f^*\omega)$
Definition 0.1.40 Let \( \omega \) be a differential \( k \)-form. Then \( \omega \) is called closed if \( d\omega = 0 \), and \( \omega \) is called exact if there exists a differential \( (k-1) \)-form \( \eta \) such that \( d\eta = \omega \).

Remark 0.1.41 The previous theorem implies that all exact forms are closed.

Exercise 0.1.42

i. If \( \omega = P \, dx + Q \, dy + R \, dz \) is a differential 1-form on \( \mathbb{R}^3 \), then compute \( d\omega \).

ii. If \( \omega = f_1 \, dx \wedge dy + f_2 \, dx \wedge dz + f_3 \, dy \wedge dz \) is a differential 2-form on \( \mathbb{R}^3 \), then compute \( d\omega \).

Exercise 0.1.43

i. If \( \omega = P \, dx + Q \, dy \) is a differential 1-form on \( \mathbb{R}^2 \), then compute \( d\omega \) (and simplify!).

ii. If \( \omega \) is as above with \( P, Q \) smooth functions on \( \mathbb{R}^2 \), and \( \omega \) is closed, show there is some \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( df = \omega \), that is, \( \omega \) is exact.

iii. If \( \omega = -\frac{y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy \) on \( \mathbb{R}^2 \setminus \{(0,0)\} \), show there does not exist \( f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \) such that \( df = \omega \).

In the next several statements, we consider restrictions under which closed forms are exact.

Definition 0.1.44 We say that a region \( A \subset \mathbb{R}^n \) is star-shaped (with respect to \( a \)) provided that, given any \( x \in A \), the line segment \( \{tx + (1-t)a \mid 0 \leq t \leq 1\} \) is also contained in \( A \).

Exercise 0.1.45 Is every star-shaped region convex? Is every star-shaped region connected? Explain.

Definition 0.1.46 Let \( A \subset \mathbb{R}^n \) be an open region that is star-shaped with respect to 0. Let

\[
\omega = \sum_{i_1 < \cdots < i_\ell} \omega_{i_1 \ldots i_\ell} \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}
\]

be a differential \( \ell \)-form on \( A \). Then we define the differential \((\ell-1)\)-form \( I\omega \) on \( A \) by

\[
I\omega(x) = \sum_{i_1 < \cdots < i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left( \int_0^1 t^{\ell-1} \omega_{i_1 \ldots i_\ell}(tx) \, dt \right) x_{i_\alpha} \, dx_{i_1} \wedge \cdots \wedge \hat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_\ell},
\]

where the \( \hat{\cdot} \) indicates that the \( dx_{i_\alpha} \) term is omitted.

Theorem 0.1.47 (Poincaré Lemma)

Let \( A \subset \mathbb{R}^n \) be an open region which is star-shaped with respect to 0. Then every closed form on \( A \) is exact. (Hint: Show that \( d(I\omega) + I(d\omega) = \omega \).)

Exercise 0.1.48 Let \( U \subset \mathbb{R}^n \), and let \( f : U \to \mathbb{R}^n \) be differentiable with differentiable inverse \( f^{-1} : f(U) \to \mathbb{R}^n \). Show that if every closed form on \( U \) is exact, then every closed form on \( f(U) \) is exact.

Definition 0.1.49 The standard \( n \)-cube in \( \mathbb{R}^n \) is the \( n \)-fold Cartesian product \( [0,1]^n = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n \). A singular \( n \)-cube in \( A \subset \mathbb{R}^m \) is a continuous function \( c : [0,1]^n \to A \). Note that the standard \( n \)-cube may be viewed as a singular \( n \)-cube in \( \mathbb{R}^n \) under the natural map \( I : [0,1]^n \to \mathbb{R}^n \) given by \( I(x) = x \). Finally, we note that we may extend the definition to \( n = 0 \) by viewing a singular 0-cube in \( A \) as a point in \( A \).

Definition 0.1.50 Let \( A \subset \mathbb{R}^m \). An \( n \)-chain in \( A \) is a formal sum \( \sum_{i=1}^k \alpha_i \cdot c_i \), where each \( c_i \) is a singular \( n \)-cube, and each \( \alpha_i \) is an integer. In other words, if \( \mathcal{S} \) is the set of all singular \( n \)-cubes in \( A \), then an \( n \)-chain is a function \( f : \mathcal{S} \to \mathbb{Z} \) such that \( f(c) = 0 \) for all but finitely many \( c \in \mathcal{S} \).
**Exercise 0.1.51** Show that if \( f \) and \( g \) are \( n \)-chains in \( A \) and \( a \in \mathbb{Z} \), then \( f + g \) and \( af \) are also \( n \)-chains in \( A \) under the natural definitions \((f + g)(c) = f(c) + g(c)\) and \((af)(c) = a \cdot f(c)\).

**Definition 0.1.52** If \( c \) is a singular \( n \)-chain in \( A \), then we define its boundary to be the \((n-1)\)-chain \( \partial c \) as follows. We first make the following definitions for the standard \( n \)-cube \( I^n \). For each \( 1 \leq i \leq n \), we define two singular \((n-1)\)-cubes \( I^n_{(i,0)} \) and \( I^n_{(i,1)} \) by:

\[
I^n_{(i,0)}(x) = I^n(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)
\]

\[
I^n_{(i,1)}(x) = I^n(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_n)
\]

Then

\[
\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.
\]

For a general singular \( n \)-cube \( c \), we define \( c_{(i,\alpha)} = c \circ (I^n_{(i,\alpha)}) \) and

\[
\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}.
\]

Finally, if \( c = \sum a_i c_i \) is an \( n \)-chain, then we define its boundary as

\[
\partial c = \partial(\sum a_1 c_i) = \sum a_i \partial(c_i).
\]

**Theorem 0.1.53** If \( c \) is an \( n \)-chain, then \( \partial(\partial c) = 0 \). (This idea is often abbreviated by \( \partial^2 = 0 \).)

**Exercise 0.1.54** Let \( A = \mathbb{R}^2 \setminus \{(0,0)\} \), and define \( c : [0,1] \to A \) by \( c(t) = (\cos 2\pi t, \sin 2\pi t) \). Show that \( \partial c = 0 \) but that there is no 2-chain \( c' \) in \( A \) such that \( \partial c' = c \). (Hint: Use Stokes’ Theorem.)

**Definition 0.1.55** (Integration on chains)

Let \( A \subset \mathbb{R}^n \). If \( \omega \) is a differential \( k \)-form \((k \geq 1)\) on \( A \) and \( c \) is a singular \( k \)-cube in \( A \), then we define the integral of \( \omega \) over \( c \) as:

\[
\int_c \omega = \int_{[0,1]^k} c^* \omega.
\]

If \( k = 0 \), then the differential 0-form \( \omega \) is a function and the singular 0-cube \( c : \{0\} \to A \) is a point, so we define

\[
\int_c \omega = \omega(c(0)).
\]

Finally, if \( c = \sum a_i c_i \) is a \( k \)-chain, we define the integral of \( \omega \) over \( c \) as:

\[
\int_c \omega = \sum a_i \int_{c_i} \omega.
\]

**Theorem 0.1.56** (Stokes’ Theorem)

Let \( A \subset \mathbb{R}^n \) be an open set. If \( \omega \) is a differential \((k-1)\)-form on \( A \) and \( c \) is a \( k \)-chain in \( A \), then

\[
\int_c d\omega = \int_{\partial c} \omega.
\]