Analysis in $\mathbb{R}^n$

Math 205, Section 30

Spring Quarter 2008

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Integration

0.1 Differential Forms

Definition 0.1.1 Let V be a vector space over $\mathbb{R}$. Then $V^k = V \times \cdots \times V = \{(v_1, \ldots, v_k) \mid v_i \in V, 1 \leq i \leq k\}$.

Definition 0.1.2 If V is a vector space over $\mathbb{R}$, a function $T : V^k \to \mathbb{R}$ is called *multi-linear* (or sometimes *k-linear*) provided that, for each $i = 1, \ldots, k$, we have:

$$\begin{aligned} T(v_1, \ldots, v_i + v_i', \ldots, v_k) &= T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k) \\ T(v_1, \ldots, \alpha v_i, \ldots, v_k) &= \alpha T(v_1, \ldots, v_i, \ldots, v_k) \end{aligned}$$

for all $v_j \in V$ and all $\alpha \in \mathbb{R}$.

Definition 0.1.3 The space $\mathcal{T}^k(V) = \{T : V^k \to \mathbb{R} \mid T \text{ is } k\text{-linear}\}$ is called the collection of *k-tensors* on V.

Definition 0.1.4 Let V be a vector space over $\mathbb{R}$. The *dual space* is the collection of 1-tensors and is denoted $V^* = \mathcal{T}^1(V) = \{T : V \to \mathbb{R} \mid T \text{ is linear}\}$.

Exercise 0.1.5

i. Show that V^* is a vector space over $\mathbb{R}$.

ii. If V is finite-dimensional, find $dim(V^*)$.

iii. If V is finite-dimensional, show that there is a natural isomorphism $V \xrightarrow{\sim} (V^*)^*$.

Exercise 0.1.6 Show that $\mathcal{T}^k(V)$ is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined as follows:

$$\begin{aligned} (T_1 + T_2)(v_1, \ldots, v_k) &= T_1(v_1, \ldots, v_k) + T_2(v_1, \ldots, v_k) \\ (\alpha T)(v_1, \ldots, v_k) &= \alpha T(v_1, \ldots, v_k) \end{aligned}$$

Definition 0.1.7 We define the *tensor product* of two tensors as follows. If $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^\ell(V)$, then $S \otimes T \in \mathcal{T}^{k+\ell}(V)$, where

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+\ell}).$$

Exercise 0.1.8 If S, S_1, S_2, T, T_1, T_2 and U are tensors and $\alpha \in \mathbb{R}$, show the following:

i. $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$

ii. $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$

iii. $(\alpha S) \otimes T = \alpha(S \otimes T) = S \otimes (\alpha T)$

iv. $S \otimes (T \otimes U) = (S \otimes T) \otimes U$

Theorem 0.1.9 If $dim(V) = n$, then the dimension of $\mathcal{T}^k(V)$ is n^k.

(Hint: Construct tensors in $\mathcal{T}^k(V)$ as tensor products of basis elements of the dual space V^*.)

Definition 0.1.10 We say that a tensor $T \in \mathcal{T}^k(V)$ is *symmetric* provided that:

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for every $v_1, \ldots, v_k \in V$ and all $1 \leq i < j \leq k$. The collection of symmetric k-tensors in $\mathcal{T}^k(V)$ is denoted $\mathcal{S}^k(V)$.

We say that T is *alternating* provided that:

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for every $v_1, \ldots, v_k \in V$ and all $1 \leq i < j \leq k$. The collection of alternating k-tensors in $\mathcal{T}^k(V)$ is denoted $\Lambda^k(V)$.

Exercise 0.1.11 Show that the standard dot product on $V = \mathbb{R}^n$ is a symmetric 2-tensor. Here, we realize the dot product as $T : V \times V \to \mathbb{R}$ given by $T(v_1, v_2) = \langle v_1, v_2 \rangle$.

Exercise 0.1.12 Show that the determinant is an alternating n-tensor on $V = \mathbb{R}^n$. Here, we realize the determinant as $D : V^n \to \mathbb{R}$ given by $D(v_1, \ldots, v_n) = \det \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \\ | & \cdots & | \end{bmatrix}$.

Exercise 0.1.13 Show that $\mathcal{S}^2(V) \oplus \Lambda^2(V) = \mathcal{T}^2(V)$ by proving the following:

i. $\mathcal{S}^2(V)$ and $\Lambda^2(V)$ are subspaces of $\mathcal{T}^2(V)$.

ii. $\mathcal{S}^2(V) \cap \Lambda^2(V) = \{0\}$

iii. $\mathcal{S}^2(V) + \Lambda^2(V) = \mathcal{T}^2(V)$
(This means that any $T \in \mathcal{T}^2(V)$ may be written as $T = S + A$ with $S \in \mathcal{S}^2(V)$ and $A \in \Lambda^2(V)$.)

Definition 0.1.14 If $T \in \mathcal{T}^k(V)$, we define the following operators *Sym* and *Alt* as follows:

$$Sym(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_n} T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

$$Alt(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Exercise 0.1.15 Prove the following properties of *Sym* and *Alt*:

i. If $T \in \mathcal{T}^k(V)$, then $Sym(T) \in \mathcal{S}^k(V)$.

ii. If $T \in \mathcal{T}^k(V)$, then $Alt(T) \in \Lambda^k(V)$.

iii. If $T \in \mathcal{S}^k(V)$, then $Sym(T) = T$.

iv. If $T \in \Lambda^k(V)$, then $Alt(T) = T$.

Exercise 0.1.16 If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, show by example that it is not necessarily the case that $\omega \otimes \eta \in \Lambda^{k+\ell}(V)$.

Definition 0.1.17 If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, we define their *wedge product* as follows:

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!} Alt(\omega \otimes \eta).$$

It is clear from earlier exercises that $\omega \wedge \eta \in \Lambda^{k+\ell}(V)$.

Exercise 0.1.18 If ω is a k-tensor, η is an ℓ-tensor, and α is a scalar, prove the following elementary facts about the wedge product:

i. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$

ii. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$

iii. $(\alpha\omega) \wedge \eta = \alpha(\omega \wedge \eta) = \omega \wedge (\alpha\eta)$

iv. $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$

Theorem 0.1.19

i. If $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^\ell(V)$ and $Alt(S) = 0$, then

$$Alt(S \otimes T) = Alt(T \otimes S) = 0.$$

ii. If $\omega \in \mathcal{T}^k(V)$, $\eta \in \mathcal{T}^\ell(V)$, and $\theta \in \mathcal{T}^m(V)$, then

$$Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta)).$$

iii. If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$, and $\theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+\ell+m)!}{k!\ell!m!} Alt(\omega \otimes \eta \otimes \theta).$$

The implication of part iii. is that the wedge product is associative, and we may simply express any of the three terms as $\omega \wedge \eta \wedge \theta$.

Theorem 0.1.20 Let$V$ be a finite-dimensional vector space with $dim(V) = n$. If $\{\varphi_1, \ldots, \varphi_n\}$ is a basis for V^*, then the set of k-tensors $\{\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ is a basis for $\Lambda^k(V)$, which thus has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Corollary 0.1.21 $dim(\Lambda^n(\mathbb{R}^n)) = 1$, and the determinant D (see 0.1.12) is a basis element.

Theorem 0.1.22 Let V be a finite-dimensional vector space over $\mathbb{R}$ with basis $\{v_1, \ldots, v_n\}$, and let $\omega \in \Lambda^n(V)$. If $\{w_1, \ldots, w_n\} \subset V$ with each $w_i = \sum_{j=1}^n a_{ij} v_j$, then

$$\omega(w_1, \ldots, w_n) = det(a_{ij}) \cdot \omega(v_1, \ldots, v_n).$$

Definition 0.1.23 For a fixed $p \in \mathbb{R}^n$, the *tangent space* to $\mathbb{R}^n$ at p is the set $(\mathbb{R}^n)_p = \{(p, v) \mid v \in \mathbb{R}^n\}$. This is a vector space with addition and scalar multiplication defined by $(p, v_1) + (p, v_2) = (p, v_1 + v_2)$ and $\alpha(p, v) = (p, \alpha v)$, respectively. This vector space has an inner product $\langle \cdot, \cdot \rangle_p : (\mathbb{R}^n)_p \times (\mathbb{R}^n)_p \to \mathbb{R}$ defined in the natural way by $\langle (p, v), (p, w) \rangle_p = \langle v, w \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product. Often, for simplicity, we will denote the point (p, v) by v_p.

Definition 0.1.24 A *vector field* on $\mathbb{R}^n$ is a function F that assigns to every $p \in \mathbb{R}^n$ a vector $F(p) \in (\mathbb{R}^n)_p$. If $(e_1)_p, \ldots, (e_n)_p$ is the standard basis in $(\mathbb{R}^n)_p$, then any $F(p)$ may be expressed as $F(p) = F^1(p) \cdot (e_1)_p + \cdots F^n(p) \cdot (e_n)_p$ for some scalars $F^1(p), \ldots, F^n(p)$. The function F is said to be continuous, differentiable, etc. provided that each of its component functions F^i is.

Definition 0.1.25 A *differential k-form* on $\mathbb{R}^n$ is a function ω that assigns to every $p \in \mathbb{R}^n$ a k-tensor $\omega(p) \in \Lambda^k((\mathbb{R}^n)_p)$. If $(e_1)_p, \ldots, (e_n)_p$ is the standard basis in each $(\mathbb{R}^n)_p$ and $\varphi_1(p), \ldots, \varphi_n(p)$ is the dual basis, then there exist functions $\omega_{i_1,\ldots,i_k} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k}(p) \cdot \varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p).$$

The function ω is said to be continuous, differentiable, C^∞, etc. provided that each of the functions $\omega_{i_1,\ldots,i_k}$ is.

Remark 0.1.26

1. If ω and η are differential k-forms, then we define $\omega + \eta$ in the obvious way.

2. If ω is a differential k-form and η is a differential ℓ-form, then we define $\omega \wedge \eta$ to be a differential $(k + \ell)$-form in the obvious way.

3. If ω is a differential k-form and f is a function, then we define $f \cdot \omega$ in the obvious way. In fact, we may think of f as a differential 0-form, and if we do, then we may denote the product above by $f \cdot \omega = f \wedge \omega$.

Exercise 0.1.27 If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$.
(Don't write anything down, but think carefully about what $Df(p)$ is.)

Exercise 0.1.28 The map df defined by $df(p)(v_p) = Df(p)(v)$ is a differential 1-form.

Definition 0.1.29 Let $x_i : \mathbb{R}^n \to \mathbb{R}$ denote the i-th coordinate projection. That is, if $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then $x_i(v) = v_i$.

Exercise 0.1.30 Show that $\{dx_1(p), \ldots, dx_n(p)\}$ is the dual basis to $\{(e_1)_p, \ldots, (e_n)_p\}$.

Remark 0.1.31 The previous exercise implies that every differential k-form ω may be written as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for some functions $\omega_{i_1,\ldots,i_k}$.

Theorem 0.1.32 If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then $df = D_1 f \cdot dx_1 + \cdots D_n f \cdot dx_n$.

Definition 0.1.33 Suppose $f : V \to W$ is a linear map between finite-dimensional real vector spaces. Then, for any $k \in \mathbb{N}$, we have an induced map $f^* : \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ defined by

$$f^*(T)(v_1, \dots, v_k) = T(f(v_1), \dots, f(v_k)),$$

for any $T \in \mathcal{T}^k(W)$.

Exercise 0.1.34 Assume $f : V \to W$ is a linear map of finite-dimensional real vector spaces.

i. If S and T are tensors, show that $f^*(S \otimes T) = f^*S \otimes f^*T$.

ii. If ω and η are alternating tensors, show that $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Definition 0.1.35 Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable. Then, for each $p \in \mathbb{R}^n$, we have a linear transformation $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$. This yields a map $f_* : (\mathbb{R}^n)_p \to (\mathbb{R}^m)_{f(p)}$ defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)}.$$

As in Definition 0.1.33 above, this linear map induces a linear map $f^* : \Lambda^k((\mathbb{R}^m)_{f(p)}) \to \Lambda^k((\mathbb{R}^n)_p)$ as follows. If ω is a differential k-form on $\mathbb{R}^m$, then $f^*\omega$ is a differential k-form on $\mathbb{R}^n$ given by

$$(f^*\omega)(p) = f^*(\omega(f(p))).$$

In other words, if $v_1, \dots, v_k \in (\mathbb{R}^n)_p$, then

$$f^*\omega(p)(v_1, \dots, v_k) = \omega(f(p))(f_*(v_1), \dots, f_*(v_k)).$$

Theorem 0.1.36 Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Then:

i. $f^*(dx_i) = \sum_{j=1}^n D_j f_i \cdot dx_j$

ii. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

iii. $f^*(g \cdot \omega) = (g \circ f) \cdot f^*(\omega)$

iv. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

Theorem 0.1.37 If $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, then

$$f^*(g \cdot dx_1 \wedge \cdots \wedge dx_n) = (g \circ f)(detDf)dx_1 \wedge \cdots \wedge dx_n.$$

Definition 0.1.38 We define an operator d on differential forms as follows. If ω is a differential k-form defined by

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \dots, i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

then $d\omega$ is a differential $(k+1)$-form, called the *differential* of ω, defined by

$$d\omega = \sum_{i_1 < \cdots < i_k} d\omega_{i_1, \dots, i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^n D_j(\omega_{i_1, \dots, i_k}) \cdot dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Theorem 0.1.39 Let ω and θ be differentiable k-forms, η a differentiable ℓ-form, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a differentiable function. Then:

i. $d(\omega + \theta) = d\omega + d\theta$

ii. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{k\ell}\omega \wedge d\eta$

iii. $d(d\omega) = 0$

iv. $f^*(d\omega) = d(f^*\omega)$

Definition 0.1.40 Let ω be a differential k-form. Then ω is called *closed* if $d\omega = 0$, and ω is called *exact* if there exists a differential $(k-1)$-form η such that $d\eta = \omega$.

Remark 0.1.41 The previous theorem implies that all exact forms are closed.

Exercise 0.1.42

i. If $\omega = P\,dx + Q\,dy + R\,dz$ is a differential 1-form on $\mathbb{R}^3$, then compute $d\omega$.

ii. If $\omega = f_3\,dx \wedge dy + f_2\,dx \wedge dz + f_1\,dy \wedge dz$ is a differential 2-form on $\mathbb{R}^3$, then compute $d\omega$.

Exercise 0.1.43

i. If $\omega = P\,dx + Q\,dy$ is a differential 1-form on $\mathbb{R}^2$, then compute $d\omega$ (and simplify!).

ii. If ω is as above with P, Q smooth functions on $\mathbb{R}^2$, and ω is closed, show there is some $f : \mathbb{R}^2 \to \mathbb{R}$ such that $df = \omega$, that is, ω is exact.

iii. If $\omega = -\dfrac{y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy$ on $\mathbb{R}^2 \setminus \{(0,0)\}$, show there does not exist $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ such that $df = \omega$.

In the next several statements, we consider restrictions under which closed forms are exact.

Definition 0.1.44 We say that a region $A \subset \mathbb{R}^n$ is *star-shaped (with respect to a)* provided that, given any $x \in A$, the line segment $\{tx + (1-t)a \mid 0 \le t \le 1\}$ is also contained in A.

Exercise 0.1.45 Is every star-shaped region convex? Is every star-shaped region connected? Explain.

Definition 0.1.46 Let $A \subset \mathbb{R}^n$ be an open region that is star-shaped with respect to 0. Let

$$\omega = \sum_{i_1 < \cdots < i_\ell} \omega_{i_1,\ldots,i_\ell}\, dx_{x_1} \wedge \cdots \wedge dx_{i_\ell}$$

be a differential ℓ-form on A. Then we define the differential $(\ell-1)$-form $I\omega$ on A by

$$I\omega(x) = \sum_{i_1<\cdots<i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^{\ell-1} \omega_{i_1,\ldots,i_\ell}(tx)\, dt \right) x_{i_\alpha}\, dx_{x_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_\ell},$$

where the $\widehat{\ }$ indicates that the dx_{i_α} term is omitted.

Theorem 0.1.47 (Poincaré Lemma)
Let $A \subset \mathbb{R}^n$ be an open region which is star-shaped with respect to 0. Then every closed form on A is exact. (Hint: Show that $d(I\omega) + I(d\omega) = \omega$.)

Exercise 0.1.48 Let $U \subset \mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be differentiable with differentiable inverse $f^{-1} : f(U) \to \mathbb{R}^n$. Show that if every closed form on U is exact, then every closed form on $f(U)$ is exact.

Definition 0.1.49 The *standard n-cube* in $\mathbb{R}^n$ is the n-fold Cartesian product $[0,1]^n = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n$. A *singular n-cube* in $A \subset \mathbb{R}^m$ is a continuous function $c : [0,1]^n \to A$. Note that the standard n-cube may be viewed as a singular n-cube in $\mathbb{R}^n$ under the natural map $I : [0,1]^n \to \mathbb{R}^n$ given by $I(x) = x$. Finally, we note that we may extend the definition to $n = 0$ by viewing a singular 0-cube in A as a point in A.

Definition 0.1.50 Let $A \subset \mathbb{R}^m$. An *n-chain* in A is a formal sum $\sum_{i=1}^k \alpha_i \cdot c_i$, where each c_i is a singular n-cube, and each α_i is an integer. In other words, if $\mathcal{S}$ is the set of all singular n-cubes in A , then an n-chain is a function $f : \mathcal{S} \to \mathbb{Z}$ such that $f(c) = 0$ for all but finitely many $c \in \mathcal{S}$.

Exercise 0.1.51 Show that if f and g are n-chains in A and $a \in \mathbb{Z}$, then $f+g$ and af are also n-chains in A under the natural definitions $(f+g)(c) = f(c) + g(c)$ and $(af)(c) = a \cdot f(c)$.

Definition 0.1.52 If c is a singular n-chain in A, then we define its *boundary* to be the $(n-1)$-chain ∂c as follows. We first make the following definitions for the standard n-cube I^n. For each $1 \leq i \leq n$, we define two singular $(n-1)$-cubes $I^n_{(i,0)}$ and $I^n_{(i,1)}$ by:

$$I^n_{(i,0)}(x) = I^n(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

$$I^n_{(i,1)}(x) = I^n(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}).$$

Then

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.$$

For a general singular n-cube c, we define $c_{(i,\alpha)} = c \circ (I^n_{(i,\alpha)})$ and

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}.$$

Finally, if $c = \sum a_i c_i$ is an n-chain, then we define its boundary as

$$\partial c = \partial(\sum a_1 c_i) = \sum a_i \partial(c_i).$$

Theorem 0.1.53 If c is an n-chain, then $\partial(\partial c) = 0$. (This idea is often abbreviated by $\partial^2 = 0$.)

Exercise 0.1.54 Let $A = \mathbb{R}^2 \setminus \{(0,0)\}$, and define $c : [0,1] \to A$ by $c(t) = (\cos 2\pi t, \sin 2\pi t)$. Show that $\partial c = 0$ but that there is no 2-chain c' in A such that $\partial c' = c$. (Hint: Use Stokes' Theorem.)

Definition 0.1.55 (Integration on chains)

Let $A \subset \mathbb{R}^n$. If ω is a differential k-form ($k \geq 1$) on A and c is a singular k-cube ini A, then we define the integral of ω over c as:

$$\int_c \omega = \int_{[0,1]^k} c^* \omega.$$

If $k = 0$, then the differential 0-form ω is a function and the singular 0-cube $c : \{0\} \to A$ is a point, so we define

$$\int_c \omega = \omega(c(0)).$$

Finally, if $c = \sum a_i c_i$ is a k-chain, we define the integral of ω over c as:

$$\int_c \omega = \sum a_i \int_{c_i} \omega.$$

Theorem 0.1.56 (Stokes' Theorem)

Let $A \subset \mathbb{R}^n$ be an open set. If ω is a differential $(k-1)$-form on A and c is a k-chain in A, then

$$\int_c d\omega = \int_{\partial c} \omega.$$