Some useful facts

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I think these are some facts that make life easier if you know them.

0. The chain rule exists.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$. If you want to compute the directional derivative $D_v f$ at points where the function is differentiable, use 2.20, DON'T use the definition. That is, compute the gradient $\nabla f$ and “dot” it with the vector $v$.

2. If you are given an explicit expression $f$ in terms of $x$ and want to show that it is differentiable at some point, try computing the partial derivatives first and see if they are continuous. If so, then $f$ is differentiable by 2.23. DON'T try to guess an explicit linear map (transformation) and show that it satisfies the definition of the derivative.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. As in the one variable case, the derivative does not change if we add a constant vector $v \in \mathbb{R}^m$ to $f$.

   Proof. We define $g(x) = f(x) + v$. Then $Dg(x) = D(f + v)(x) = Df(x) + Dv(x) = Df(x) + 0$ since $Dv = 0$ because $v$ is a constant (vector-valued) function. \hfill \Box

4. Let $f$ be as in 3. and $w \in \mathbb{R}^n$. If $h(x) = f(x + w)$, then $Dh(x) = Df(x + w)$, or equivalently $Dh(x - w) = Df(x)$.

   Proof. We define $k(x) = x + w$. Then $h = f \circ k$. By the chain rule, $Dh(x) = Df(k(x)) \circ Dk(x) = Df(x + w) \circ I = Df(x + w)$ \hfill \Box

   (How does this apply to Jahnavi’s proof (some of the WLOG part) of theorem 4.1?)
5 If \( l : \mathbb{R}^n \to \mathbb{R}^n \) is a linear map, and \( h := f \circ l \), then \( Df(x) = Df(l(x)) \circ l(x) \).

Proof. (Sketch) Since \( l \) is a linear map, \( Dl = l \). Now apply the chain rule. \( \square \)

(This is why in Jahnavi’s proof of theorem 4.1 we can assume that \( \nabla g_1, \ldots, \nabla g_k \) span the subspace spanned by the first \( k \) vectors of the standard basis. This is because if they span different subspace, we can compose \( g_i \)'s with a rotation matrix and take this subspace to the standard one.)

6. Let \( V \) be a \( k \)-dim vector subspace of \( \mathbb{R}^n \) \((k \leq n)\) with an inner product. Then \( V^\perp := \{ u \in \mathbb{R}^n | u \cdot v = 0 \quad \forall v \in V \} \) is a \((n - k)\)-dim vector subspace and \( V \cap V^\perp = \{ 0 \} \) and \( \mathbb{R}^n = V \oplus V^\perp \).

Proof. You do linear algebra. \( \square \)