Sketch solutions to questions on connectedness

Let us write $a \sim b$ iff a is path-connected to b. Note that $\sim$ is an equivalence relation.

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(i),(ii) Let B be an open or closed ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual metric. Then B is a convex set. Let $x, y \in B$. Define $p : [0,1] \to B$ by $p(t) = (1-t)x + ty$. (Note: this is a straight line path.) By convexity, p is a map into B. Since $p(0) = x$, $p(1) = y$, and p is a homeomorphism, then $x \sim y$. Since $x, y \in B$ were arbitrary, then B is path-connected. Therefore B is connected, as required.

(iii) Let $\mathrm{GL}^+(2,\mathbb{R}) = \{M \in \mathrm{GL}(2,\mathbb{R}) | \det(M) > 0\}$ and let $\mathrm{GL}^-(2,\mathbb{R}) = \{M \in \mathrm{GL}(2,\mathbb{R}) | \det(M) < 0\}$. Since the determinant function, $\det : \mathrm{M}_2(\mathbb{R}) \to \mathbb{R}$, is a continuous function, then $\det^{-1}((0,\infty)) = \mathrm{GL}^+(2,\mathbb{R})$ and $\det^{-1}((-\infty,0)) = \mathrm{GL}^-(2,\mathbb{R})$, are open sets. In fact, these sets disconnect $\mathrm{GL}(2,\mathbb{R})$. Check this.

(iv) We will show that $\mathrm{GL}(2,\mathbb{C})$ is path-connected. Let $A \in \mathrm{GL}(2,\mathbb{C})$ be arbitrary. Let U_A denote the upper triangular matrix corresponding to A. That is, U_A is constructed by performing elementary row operations on A until all elements below the diagonal are zero. It can be shown that for any $M \in \mathrm{GL}(2,\mathbb{C})$, $M \sim M'$, where M' is the result of applying one elementary row operation to M. It follows that $A \sim U_A$. Since $\mathbb{C} \setminus \{0\}$ is path-connected, then $U_A \sim I_n$, where $I_n = \mathrm{diag}(1,\ldots,1)$. (Create a path from U_A to I_n by creating paths from each of the corresponding matrix elements. Use straight-line paths on non-diagonal elements and paths that do not pass through zero on the diagonal elements. Such a path will alway remain in $\mathrm{GL}(2,\mathbb{C})$ since the determinant of an upper triangular matrix is equal to the product of the diagonal entries.) By transitivity, $A \sim I_n$. Since $A \in \mathrm{GL}(2,\mathbb{C})$ was arbitrary, it follows that $\mathrm{GL}(2,\mathbb{C})$ is path-connected and therefore connected.

4.5.39 $\mathrm{O}(n) = \{M \in \mathrm{GL}(n,\mathbb{R}) | MM^T = I_n\}$, where I_n denotes the identity matrix. Let $M \in \mathrm{O}(n)$. Since $1 = \det(I_n) = \det(MM^T) = \det(M)\det(M^T) = \det(M)\det(M)$, then $\det(M) \in \{1,-1\}$. Let $\mathrm{O}(n)^+ = \{M \in \mathrm{O}(n) | \det(M) = 1\} = \mathrm{GL}^+(n,R) \cap \mathrm{O}(n)$ and let $\mathrm{O}(n)^- = \{M \in \mathrm{O}(n) | \det(M) = -1\} = \mathrm{GL}^-(n,\mathbb{R}) \cap \mathrm{O}(n)$. Then $\mathrm{O}(n)^+$ and $\mathrm{O}(n)^{-1}$ are open in the subspace topology and they disconnect $\mathrm{O}(n)$. Check this.
$\mathrm{SO}(n) = \{M \in \mathrm{O}(n) | \det(M) = 1\}$. Let $M \in \mathrm{SO}(n)$. Then $M = UDU^T$ where $U \in \mathrm{SO}(n)$ and $D = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. Without loss of generality, $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$. Since $\det(M) = 1$, then there must be an even number of λ_is that are negative, say $\lambda_1,\ldots,\lambda_{2k}$. Since $\mathbb{R}^+$ and $\mathbb{R}^-$ are path-connected, then $D \sim \mathrm{diag}(-1,\ldots,-1,1,\ldots,1)$. It can be shown that $I_2 \sim -I_2$ by showing that $I_2 \sim N$ and $N \sim -I_2$, where

$$N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It follows that $D \sim I_n$ and so $M \sim UI_nU^T = I_n$. Thus, $\mathrm{SO}(n)$ is path-connected and therefore connected.