Math 207, Section 31: Honors Analysis I
Autumn Quarter 2009
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Homework 1, Version 1
Due: Monday, October 5, 2009

From Homework 1 onward, starred (*) problems are to be considered "moral" homework, which means that you are responsible for the material, but the problems do not need to be written up and submitted for grading.

1. (*) Read Sally, Chapter 1.
2. (*) Read Kolmogorov and Fomin, Chapter 1, especially Sections 1 and 2.
3. Sally, Section 1.5, Exercises (*) 1.5.1(2), (*) 1.5.2, (*)1.5.3, 1.5.6, 1.5.7, 1.5.8, 1.5.9, and 1.5.10.
4. Sally, Section 1.6, Exercises 1.6.9, 1.6.12, 1.6.15, 1.6.16, 1.6.17 and 1.6.24.
5. Do the exercises in Sally, Chapter 1, Project 10.1.
6. Do the exercises in Sally, Chapter 1, Project 10.3.
7. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is never an integer for $n \geq 2$.
8. Assume the following definitions:

   Definition. The natural numbers $\mathbb{N}$ are (up to isomorphism) the unique set with a successor function $S : \mathbb{N} \to \mathbb{N}$ and distinguished element 1 satisfying:

   1) $1 \notin \mathrm{Im}(S)$
   2) S is one-to-one
   3) If $M \subset \mathbb{N}$ satisfies (i) and (ii) below, then $M = \mathbb{N}$:
      (i) $1 \in M$
      (ii) if $n \in M$, then $S(n) \in M$,

   Definition Addition in $\mathbb{N}$ is defined as follows. Fix $n \in \mathbb{N}$. Then:

   1) $n + 1 = S(n)$,
   2) and, inductively, if $n + m$ is defined for $m \in \mathbb{N}$, then $n + (m + 1) = S(n + m)$.

   Definition Multiplication in $\mathbb{N}$ is defined as follows. Fix $n \in \mathbb{N}$. Then:

   1) $n \cdot 1 = n$,
   2) and, inductively, if $n \cdot m$ is defined for $m \in \mathbb{N}$, then $n \cdot (m + 1) = (n \cdot m) + n$.

   Definition Order in $\mathbb{N}$ is defined by:

   $n < m$ provided that there is some $d \in \mathbb{N}$ such that $n + d = m$.

   Show that $\mathbb{N}$ satisfies the commutativity of addition and multiplication, the associativity of multiplication, the distributivity of multiplication over addition, the rules of order (O1–O4 in Sally's notation).

9. Assume the following definitions:

Definition. The integers $\mathbb{Z}$ are the set of equivalence classes of ordered pairs of natural numbers as follows:

$$\mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{N}\}/\sim$$

where $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

(a) Show that addition defined by $[(a, b)] + [(c, d)] = [(a + c, b + d)]$ is well-defined.

(b) Show that addition, as defined above, is associative and commutative, that $\mathbb{Z}$ has an additive identity, and that every element in $\mathbb{Z}$ has an additive inverse.

(c) Give a definition of multiplication, and show that it is well-defined.

(d) Show that multiplication, as defined above, is associative and commutative, and that $\mathbb{Z}$ has an multiplicative identity.

(e) Show that, as defined above, multiplication is distributive over addition.

(f) Show that the order relation defined by $[(a, b)] < [(c, d)]$ if $b + c < a + d$ is well-defined.

(g) Show that order, as defined above, satisfies the rules of order.