Math 208, Section 31: Honors Analysis II
Winter Quarter 2010
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Homework 1, Version 2
Due: Monday, January 11, 2010

To remind you, starred (*) problems are to be considered "moral" homework, which means that you are responsible for the material, but the problems do not need to be written up and submitted for grading.

1. (*) Read a good source on Differentiation of Functions of a single real variable. Perhaps Rudin (Chapter 5), Edwards (Chapter 2), or Lang (Chapter 3).
2. Read Sally, Section 5.2, and do exercises (*) 5.2.3, 5.2.6, 5.2.7, 5.2.10, 5.2.11, and (*) 5.2.13.
3. (*) Prove the Product and Quotient Rules for two functions $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ that are differentiable at a point $c \in (a, b)$.
4. Prove the Dot Product Rule for two functions $f : (a, b) \to \mathbb{R}^k$ and $g : (a, b) \to \mathbb{R}^k$ that are differentiable at a point $c \in (a, b)$.
5. Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on the rationals and discontinuous on the irrationals.
6. Let $\alpha > 0$, and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f_\alpha(x) = \begin{cases} \frac{1}{q^\alpha} & , \text{ if } x = \frac{p}{q} \text{ in lowest terms and } p \neq 0 \\ 0 & , \text{ if } x = 0 \text{ or } x \notin \mathbb{Q}. \end{cases}$
   (a) Show that if $\alpha \geq 1$, then f_α is continuous at 0 and the irrationals but discontinuous at the non-zero rationals.
   (b) Show that if $1 \leq \alpha \leq 2$, then f_α is not differentiable at the irrationals.
   (c) For which α is f_α differentiable at 0?
   (d) Read Liouville's Theorem (1.1.28 in the notes).
   (e) Suppose $n \geq 2$ and that $x \in \mathbb{R}$ is algebraic of degree n over $\mathbb{Q}$. Show that if $\alpha > n$, then f_α is differentiable at x.
7. For each $n \in \mathbb{N}$, define a function $f_n : [0, +\infty) \to \mathbb{R}$ as follows. On $[0, 1]$, define $f_1(x) = \begin{cases} x & , \text{ if } 0 \leq x \leq \frac{1}{2} \\ 1 - x & , \text{ if } \frac{1}{2} \leq x \leq 1, \end{cases}$ and extend by periodicity via $f_1(x+1) = f_1(x)$ for all $x \geq 1$. For each $n \geq 2$, define $f_n(x) = \frac{1}{2} f_{n-1}(2x)$. Finally, define $S_m(x) = \sum_{n=1}^{m} f_n(x)$.
   (a) Show that (S_m) converges uniformly to a continuous function S.
   (b) Show that S is not differentiable at any point of $[0, +\infty)$.
8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there exists $M > 0$ such that $|f'(x)| \leq M$ for every $x \in \mathbb{R}$. Show that f is uniformly continuous.
9. Suppose $f : (a, b) \to \mathbb{R}$ and $f'(x) > 0$ for every $x \in (a, b)$.
   (a) Show that f is strictly increasing on (a, b).
   (b) If g is the inverse function to f, show that g is differentiable and that for every $x \in (a, b)$ we have:
   $$g'(f(x)) = \frac{1}{f'(x)}.$$
10. Show that if $f : (a, b) \to \mathbb{R}$ is differentiable and f' is monotonic on (a, b), then f' is continuous on (a, b).

11. Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n \in \mathbb{Q}_p$ for each n.

    We define the *formal derivative* of f to be the function $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

    Let f and g be two functions defined by power series as above, and assume x is in the domain of convergence of each. (And for part (c), assume g(x) is in the domain of convergence of f.)

    (a) Show that $(f+g)'(x) = f'(x) + g'(x)$.

    (b) Show that $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

    (c) Show that if $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$.

12. Define $exp : \mathbb{Q}_p \to \mathbb{Q}_p$ by the power series

$$exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

    (a) Show that $exp(x)$ converges on $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x| \leq \frac{1}{p}\}$ if p is odd.

    (b) Show that $exp(x)$ converges on $4\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 \mid |x| \leq \frac{1}{4}\}$ if $p = 2$.

    (c) Show that $(exp)'(x) = exp(x)$ on the domains of convergence.

13. Constuct a function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ that is differentiable, has derivative 0 everywhere, and yet is not locally constant.