Math 208, Section 31: Honors Analysis II
Winter Quarter 2010
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Homework 2, Final Version
Due: WEDNESDAY, January 20, 2010

1. (*) Read a good source on Differentiation of Functions of functions from $\mathbb{R}^n \to \mathbb{R}^m$. Perhaps the BS notes, Rudin (Chapter 9), Edwards (Chapter 2), or Lang (Chapter 15).

2. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$. Show that if f is differentiable at $x \in U$, then all of its partial derivatives exist at x and $Df(x) = \nabla f(x) = (D_1(x), \ldots, D_n f(x))$, by which we mean that $Df(x)h = \nabla f(x) \cdot h$ as a linear map with $h \in \mathbb{R}^n$.

3. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$. Show that if f is differentiable at $x \in U$, then all of its directional derivatives exist at x and if $v \in \mathbb{R}^n$ is a unit vector, then $D_v f(x) = \nabla f(x) \cdot v$.

4. Let $U \subset \mathbb{R}^n$ be a connected open set, and let $f : U \to \mathbb{R}$ be a differentiable function such that $Df \equiv 0$. Show that f is constant on U.

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^n$ be two differentiable curves, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Suppose that $p = f(s_0)$ and $q = g(t_0)$ are closer than any other pair of points on the two curves. Prove that the vector $p - q$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$.

6. Consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $f(x) = ||x||x$. Determine whether or not f is differentiable at 0. If not, why not? If so, find the first-order partial derivatives of f at 0. Do the second-order partial derivatives of f exist at 0? Explain.

7. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Show that:
   (a) (*) $\nabla(f + g) = \nabla f + \nabla g$.
   (b) (*) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.
   (c) $\nabla(f^m) = mf^{m-1}\nabla f$, for any positive integer m.
   (d) Determine a formula for $\nabla(\frac{f}{g})$ when $g(x) \neq 0$.

8. Recall the isomorphism of vector spaces $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.
   (a) Consider the determinant map $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, and find $\nabla(\det)(A)$, expressed in terms of $A = [a_{ij}]$.
   (b) Consider the function $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ given by $f(A) = A^2$. Show that $Jf_A(H) = AH + HA$.

9. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, our theorem about cross-partials applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

   With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, the usual gradient of f, we make the following definitions:

   - $||\nabla f||^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of f.
   - $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the *Laplacian* of f.

   Finally, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $g(r, \theta) = f(r\cos\theta, r\sin\theta)$.

   (a) Show that $||\nabla f||^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$.
   (b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}$.

10. if $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partials, the *Laplacian* of f is defined to be $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

With f as above, we say that f is *harmonic* on the open set $U \subset \mathbb{R}^n$ provided that $\nabla^2 f(x) = 0, \ \ \forall x \in U$.

(a) Find a (simple) condition on the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$ that makes f harmonic.

(b) Show that $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \frac{1}{||x||^{n-2}}$ is harmonic on $U = \mathbb{R}^n \setminus \{0\}$.

(c) Show that if $g : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, then $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = g(e^x \cos y, e^x \sin y)$ is also harmonic.