1. (*) Read a good source on Differentiation of Functions of functions from $\mathbb{R}^n \rightarrow \mathbb{R}^m$. Perhaps the BS notes, Rudin (Chapter 9), Edwards (Chapter 2), or Lang (Chapter 15).

2. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \rightarrow \mathbb{R}$. Show that if $f$ is differentiable at $x \in U$, then all of its partial derivatives exist at $x$ and $Df(x) = \nabla f(x) = (D_1(x), \ldots, D_n(f(x)))$, by which we mean that $Df(x)h = \nabla f(x) \cdot h$ as a linear map with $h \in \mathbb{R}^n$.

3. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \rightarrow \mathbb{R}$. Show that if $f$ is differentiable at $x \in U$, then all of its directional derivatives exist at $x$ and if $v \in \mathbb{R}^n$ is a unit vector, then $D_v f(x) = \nabla f(x) \cdot v$.

4. Let $U \subset \mathbb{R}^n$ be a connected open set, and let $f : U \rightarrow \mathbb{R}$ be a differentiable function such that $Df = 0$. Show that $f$ is constant on $U$.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}^n$ and $g : \mathbb{R} \rightarrow \mathbb{R}^n$ be two differentiable curves, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Suppose that $p = f(s_0)$ and $q = g(t_0)$ are closer than any other pair of points on the two curves. Prove that the vector $p - q$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$.

6. Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $f(x) = ||x||x$. Determine whether or not $f$ is differentiable at 0. If not, why not? If so, find the first-order partial derivatives of $f$ at 0. Do the second-order partial derivatives of $f$ exist at 0? Explain.

7. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. Show that:
   
   - (a) (*) $\nabla (f + g) = \nabla f + \nabla g$.
   - (b) (*) $\nabla (fg) = (\nabla f)g + f(\nabla g)$.
   - (c) $\nabla (f^m) = mf^{m-1}\nabla f$, for any positive integer $m$.
   - (d) Determine a formula for $\nabla (\frac{f}{g})$ when $g(x) \neq 0$.

8. Recall the isomorphism of vector spaces $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

   - (a) Consider the determinant map $\det : M_n(\mathbb{R}) \rightarrow \mathbb{R}$, and find $\nabla (\det)(A)$, expressed in terms of $A = [a_{ij}]$.
   - (b) Consider the function $f : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ given by $f(A) = A^2$. Show that $Jf_A(H) = AH + HA$.

9. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, our theorem about cross-partials applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

   With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, the usual gradient of $f$, we make the following definitions:

   - $||\nabla f||^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of $f$.
   - $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the Laplacian of $f$.

   Finally, let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $g(r, \theta) = f(r \cos \theta, r \sin \theta)$.

   - (a) Show that $||\nabla f||^2 = \frac{\partial g}{\partial r}^2 + \frac{1}{r^2} \left( \frac{\partial g}{\partial \theta} \right)^2$.
   - (b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r} \frac{\partial g}{\partial \theta}$. 

10. If \( f : \mathbb{R}^n \to \mathbb{R} \) has continuous second-order partials, the **Laplacian of** \( f \) is defined to be \( \nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \).

With \( f \) as above, we say that \( f \) is **harmonic** on the open set \( U \subset \mathbb{R}^n \) provided that \( \nabla^2 f(x) = 0, \forall x \in U \).

(a) Find a (simple) condition on the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by \( f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k \) that makes \( f \) harmonic.

(b) Show that \( f : \mathbb{R}^n \to \mathbb{R} \) defined by \( f(x) = \frac{1}{||x||^n} \) is harmonic on \( U = \mathbb{R}^n \setminus \{0\} \).

(c) Show that if \( g : \mathbb{R}^2 \to \mathbb{R} \) is harmonic, then \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f(x, y) = g(e^x \cos y, e^x \sin y) \) is also harmonic.