Math 208, Section 31: Honors Analysis II
Winter Quarter 2010
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Homework 6, Version 2
Due: Monday, February 15, 2010

1. (*) Read a good source on the Riemann Integration in $\mathbb{R}^n$. Perhaps the BS notes, volume II, Apostol (Chapter 14), Edwards (Chapter 4), or Lang (Chapter 20).

2. Read Sections 2.8 and 2.9 in the BS notes II and prove Theorem 2.9.1.

3. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$. We say that f is *absolutely integrable* on A provided that $|f|$ is integrable on A.
   (a) Prove that if f is integrable on A, then f is absolutely integrable on A.
   (b) Prove by example that the converse to (a) is false.
   (c) Prove that if f is integrable on A, then $|\int_A f| \leq \int_A |f|$.

4. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be a bounded function.
   Prove $B_\varepsilon = \{x \in A \mid o(f, x) \geq \varepsilon\}$ is compact.

5. If $A \subset \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$, show that $B = A \times \{c\}$ is a set of measure zero in $\mathbb{R}^n$.

6. Define a set $A \subset \mathbb{R}^n$ to be *negligible* if, for any $\varepsilon > 0$, there exists a finite collection $\{U_i\}_{i=1}^k$ of closed rectangles such that $A \subset \bigcup_{i=1}^n U_i$ and $\sum_{i=1}^n v(U_i) < \varepsilon$.
   (a) (*) Prove that the Cantor set is negligible.
   (b) Prove that if A is compact and has measure zero, then A is negligible.
   (c) Prove that if $A \subset \mathbb{R}^{n-1}$ is bounded and $c \in \mathbb{R}$, then $B = A \times \{c\}$ is negligible in $\mathbb{R}^n$. Prove that the boundedness condition on A is necessary for this result.
   (d) Let $A \subset \mathbb{R}^n$ be negligible, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function satisfying the *Lipschitz condition* $|f(x) - f(y)| \leq C|x - y|$ for some constant $C > 0$. Show that $f(A)$ is negligible.

7. Let $D = \{(x, y) \mid 0 \leq x \leq y \leq 1\} \subset \mathbb{R}^2$, and consider $f : D \to \mathbb{R}$ given by $f(x, y) = \sin(y^2)$. Compute $\int_D f$ by computing both iterated integrals.

8. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be positive and continuous, and suppose that

$$\int_D f = \int_0^1 \left(\int_y^{\sqrt{2-y^2}} f(x, y)\, dx \right) dy.$$

   Sketch the region D and interchange the order of integration.

9. For a function $f : [0, 1] \longrightarrow \mathbb{R}$, let $A = \{x \in [0, 1] \mid f \text{ is not differentiable at } x\}$. Find (with proof) such an f satisfying the following conditions:
   - f is continuous.
   - $f(0) = 0$
   - $f(1) = 1$
   - A is negligible. (See question 6.)
   - If $x \notin A$, then $f'(x) = 0$.

10. Let $B = \{n \in \mathbb{N} \mid \text{the decimal expansion of } n \text{ has no 8's}\}$. Decide whether the infinite series $\sum_{n\in B} \frac{1}{n}$ converges or diverges.

11. Let $A \subset \mathbb{R}^n$ be a set of measure zero, and let $f : \mathbb{R}^n \to \mathbb{R}^n$.

    (a) If f is continuous, decide (with proof) whether or not it is not necessarily true that $f(A)$ has measure zero.

    (b) If f is a homeomorphism, decide (with proof) whether or not it is not necessarily true that $f(A)$ has measure zero.

12. Let $B^n \subset \mathbb{R}^n$ be the unit ball, that is, $B^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$. Use the following exercises to show in two different ways that that the volume of B^n is given by the two formulas:

$$v(B^{2n}) = \frac{\pi^n}{n!} \quad \text{and} \quad v(B^{2n+1}) = \frac{2^{n+1}\pi^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}.$$

    (a) Let $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$, and let $Q = \{(x_3, \dots, x_n) \in \mathbb{R}^{n-2} \mid |x_i| \leq 1, \forall i\}$. Then $B^n \subset B^2 \times Q$. Let $\varphi : B^2 \times Q \to \mathbb{R}$ be the characteristic function of B^n. Note that this implies that

$$v(B^n) = \int_{B^2} \left(\int_Q \varphi(x_1, \dots, x_n)\, dx_3 \dots dx_n \right) dx_1\, dx_2.$$

        i. Show that, for a fixed $(x_1, x_2) \in \mathbb{R}^2$, the inner integral is

$$\int_Q \varphi(x_1, \dots, x_n)\, dx_3 \dots dx_n = (1 - x_1^2 - x_2^2)^{(n-2)/2} \cdot v(B^{n-2}).$$

        ii. Use polar coordinates to show that $\displaystyle\int_{B^2} (1 - x_1^2 - x_2^2)^{(n-2)/2} dx_1\, dx_2 = \frac{2\pi}{n}$.

        iii. Show that $v(B^n) = \frac{2\pi}{n} \cdot v(B^{n-2})$ for $n \geq 2$.

        iv. Prove the given formulas by induction after establishing that $v(B^1) = 2$ and $v(B^2) = \pi$.

    (b) The n-dimensional spherical coordinate change of variables is given by $T : \mathbb{R}^n \to \mathbb{R}^n$, where:

$$\begin{aligned}
x_1 &= \rho \cos\varphi_1 \\
x_2 &= \rho \sin\varphi_1 \cos\varphi_2 \\
x_3 &= \rho \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\vdots \\
x_{n-1} &= \rho \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\theta \\
x_n &= \rho \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\theta
\end{aligned}$$

    which maps the rectangle $R = \{(\phi, \varphi_1, \dots, \varphi_{n-2}, \theta) \in \mathbb{R}^n \mid \rho \in [0,1], \varphi_i \in [0, \pi], \theta \in [0, 2\pi]\}$ onto the unit ball B^n.

        i. Prove by induction that $|\det JT| = \rho^{n-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2}$.

        ii. Show that $\displaystyle v(B^n) = \int_{B^n} 1 = \int_Q |\det JT| = \frac{2\pi}{n} \prod_{k=1}^{n-2} \left[\int_0^\pi \sin^k \varphi\, d\varphi \right]$.

        iii. Show that this last formula gives the stated result.