Math 207, Section 31: Honors Analysis I
Autumn Quarter 2009
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Homework 7, Final Version
Due: Monday, November 16, 2009

1. (*) Read Kolmogorov and Fomin, Chapter 3, especially Sections 11–12.
2. (*) Read Sally, Chapter 5, especially Section 3.
3. Sally, Section 5.3, Exercises (*) 5.3.3, 5.3.6, (*) 5.3.11, 5.3.13 and 5.3.14.
   Note that 5.3.13 should be corrected to say the algebra of functions generated by the set...
4. Sally, Section 5.5, Problems 5.19 and 5.20.
5. Kolmogorov and Fomin, Section 11, Problem 4.
6. Let $X$ be a topological space. A subset $S \subset X$ is said to be *relatively compact* in $X$ if its closure $\overline{S}$ is compact in $X$. The topological space $X$ itself is said to be *locally compact* if every point $x \in X$ has a neighborhood that is relatively compact.
   (a) Decide whether or not $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are locally compact with the usual metrics.
   (b) Show that if $X$ is a compact topological space, then it is locally compact.
   (c) Show that a closed subset of a locally compact space is locally compact.
7. Consider the function $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$.
   (a) For what values of $x$ does the series converge absolutely?
   (b) On what intervals does the series converge uniformly?
   (c) Is $f$ continuous wherever it converges?
   (d) Is $f$ bounded?
8. For each $n \in \mathbb{N}$, define $f_n \in C[0,1]$ by the formula:

$$f_n(x) = \begin{cases} \sin^2 \frac{\pi}{x}, & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

   Show that $(f_n)$ converges in $C[0,1]$, but not uniformly. Use this example to show that even absolute convergence of a series $\sum f_n$ does not imply uniform convergence.
9. Let $\mathbb{F}_q$ be the finite field with $q$ elements, and let $V = (\mathbb{F}_q)^n$ be a vector space over $\mathbb{F}_q$.
   (a) How many distinct (ordered) bases does $V$ have?
   (b) For each $k = 0, 1, 2, \ldots, n$, how many distinct $k$-dimensional subspaces does $V$ have?