1. (*) Read Kolmogorov and Fomin, Chapter 7.

2. Let \( X \) be a set. A \( \sigma \)-algebra is a collection of subsets of \( X \) that contains the empty set and is closed under countable unions and complementation.
   (a) (*) Show that a \( \sigma \)-algebra is closed under set difference and countable intersection.
   (b) (*) Show that a non-empty collection of sets that contains the empty set and is closed under countable intersection and complementation is a \( \sigma \)-algebra.
   (c) Is it possible for a \( \sigma \)-algebra to be a countably infinite set? Explain.

3. Let \( X \) be a set, and let \( \mathcal{M} \) be a \( \sigma \)-algebra on \( X \). A measure on \( X \) is a function \( \mu : \mathcal{M} \to [0, \infty] \) that is countably additive.
   (a) Let \( X = \mathbb{Z} \), and let \( \mathcal{M} = \mathcal{P}(\mathbb{Z}) \). Show that \( \mu(A) = |A| \) defines a measure on \( X \). (This is called the counting measure.)
   (b) Show that if \( A, B \in \mathcal{M} \) and \( A \subset B \), then \( \mu(A) \leq \mu(B) \).
   (c) Show that if \( A_1, A_2, \ldots \in \mathcal{M} \) and \( A_1 \subset A_2 \subset \cdots \), then \( \mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \to \infty} \mu(A_i) \).

4. Show that if \( E \subset B \) and \( B \in \mathcal{L}(\mathbb{R}) \) with \( m(B) < \infty \), then \( E \in \mathcal{L}(\mathbb{R}) \) if and only if \( m(B) = m^*(E) + m^*(B \setminus E) \).

5. Show that if \( A \in \mathcal{L}(\mathbb{R}) \), then there exist \( B \in \mathcal{B}(\mathbb{R}) \) and \( N \in \mathcal{N}(\mathbb{R}) \) such that \( (A \setminus B) \cup (B \setminus A) = N \).

6. Let \( A \subset (a, b) \subset \mathbb{R} \) be a bounded set. Show that \( m((a, b)) = m_*(A) + m^*((a, b) \setminus A) \).

7. Let \( n \geq 2 \), and consider \( \mathbb{R}^n \). A set \( R = I_1 \times \cdots \times I_n \subset \mathbb{R}^n \) is called a half-open rectangle if each \( I_i = [a_i, b_i) \) for some \( a_i < b_i \), and the volume of \( R \) is defined to be \( m(R) = \prod_{i=1}^{n} (b_i - a_i) \). We define the Lebesgue outer measure of a set \( A \subset \mathbb{R}^n \) to be \( m^*(A) = \inf \{ \sum_{i \in I} m(R_i) \} \) where \( \{R_i\}_{i \in I} \) is an at most countable covering of \( A \) by half-open rectangles. We say that a set \( A \subset \mathbb{R}^n \) is Lebesgue measurable if, given any \( E \subset \mathbb{R}^n \), we have \( m^*(E) = m^*(E \cap A) + m^*(E \setminus A) \).
   (a) Show that every open set in \( \mathbb{R}^n \) may be written as the countable disjoint union of half-open rectangles.
   (b) (*) Show that \( m^*(A) \) is defined for every subset \( A \subset \mathbb{R}^n \).
   (c) Show that if \( A \subset \mathbb{R}^n \) is a rectangle, then \( A \) is Lebesgue measurable.
   (d) Show that open sets in \( \mathbb{R}^n \) are Lebesgue measurable.

8. Show that the extended real line, that is, the set \( \overline{\mathbb{R}} = [-\infty, \infty] \) endowed with the order topology, is homeomorphic to the closed unit interval \([0, 1]\) and hence is a compact Hausdorff space. (Is this space \( T_4 \)? Is it metrizable?)

9. Show that if \( f : X \to Y \) is measurable function, then for every \( B \in \mathcal{B}(Y) \), the set \( f^{-1}(B) \) is measurable.

10. Let \( f : X \to \overline{\mathbb{R}} \) be a simple function written as \( f = \sum_{i=1}^{N} \alpha_i \chi_{A_i} \), where the \( \alpha_i \) are distinct.
    (a) Show that \( f \) is measurable if each \( A_i \) is measurable.
    (b) Give an example to illustrate the necessity of the distinctness of the \( \alpha_i \) in order to show that if \( f \) is measurable, then each \( A_i \) is measurable.

11. Let \( X \) be a measure space, and let \( f : X \to [0, \infty] \) be measurable. Show that there exists a sequence of simple, measurable functions \( \{f_n\}_{n=1}^{\infty} \) that converges pointwise almost everywhere to \( f \).