Math 208, Section 31: Honors Analysis II
Winter Quarter 2010
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Homework 7, Final Version
Due: Monday, February 22, 2010

1. (*) Read Kolmogorov and Fomin, Chapter 7.
2. Let X be a set. A σ-algebra is a collection of subsets of X that contains the empty set and is closed under countable unions and complementation.
   (a) (*) Show that a σ-algebra is closed under set difference and countable intersection.
   (b) (*) Show that a non-empty collection of sets that contains the empty set and is closed under countable intersection and complementation is a σ-algebra.
   (c) Is it possible for a σ-algebra to be a countably infinite set? Explain.
3. Let X be a set, and let $\mathcal{M}$ be a σ-algebra on X. A measure on X is a function $\mu : \mathcal{M} \to [0, \infty]$ that is countably additive.
   (a) Let $X = \mathbb{Z}$, and let $\mathcal{M} = \mathcal{P}(\mathbb{Z})$. Show that $\mu(A) = |A|$ defines a measure on X. (This is called the *counting measure*.)
   (b) Show that if $A, B \in \mathcal{M}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$.
   (c) Show that if $A_1, A_2 \ldots \in \mathcal{M}$ and $A_1 \subset A_2 \subset \cdots$, then $\mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.
4. Show that if $E \subset B$ and $B \in \mathcal{L}(\mathbb{R})$ with $m(B) < \infty$, then $E \in \mathcal{L}(\mathbb{R})$ if and only if $m(B) = m^*(E) + m^*(B \setminus E)$.
5. Show that if $A \in \mathcal{L}(\mathbb{R})$, then there exist $B \in \mathcal{B}(\mathbb{R})$ and $N \in \mathcal{N}(\mathbb{R})$ such that $(A \setminus B) \cup (B \setminus A) = N$.
6. Let $A \subset (a, b) \subset \mathbb{R}$ be a bounded set. Show that $m((a, b)) = m_*(A) + m^*((a, b) \setminus A)$.
7. Let $n \geq 2$, and consider $\mathbb{R}^n$. A set $R = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ is called a *half-open rectangle* if each $I_i = [a_i, b_i)$ for some $a_i < b_i$, and the volume of R is defined to be $m(R) = \prod_{i=1}^{n}(b_i - a_i)$. We define the *Lebesgue outer measure* of a set $A \subset \mathbb{R}^n$ to be $m^*(A) = \inf\{\sum_{i\in I} m(R_i)\}$ where $\{R_i\}_{i=I}$ is an at most countable covering of A by half-open rectangles. We say that a set $A \subset \mathbb{R}^n$ is *Lebesgue measurable* if, given any $E \subset \mathbb{R}^n$, we have $m^*(E) = m^*(E \cap A) + m^*(E \setminus A)$.
   (a) Show that every open set in $\mathbb{R}^n$ may be written as the countable disjoint union of half-open rectangles.
   (b) (*) Show that $m^*(A)$ is defined for every subset $A \subset \mathbb{R}^n$.
   (c) Show that if $A \subset \mathbb{R}^n$ is a rectangle, then A is Lebesgue measurable.
   (d) Show that open sets in $\mathbb{R}^n$ are Lebesgue measurable.
8. Show that the extended real line, that is, the set $\overline{\mathbb{R}} = [-\infty, \infty]$ endowed with the order topology, is homeomorphic to the closed unit interval $[0, 1]$ and hence is a compact Hausdorff space. (Is this space T_4? Is it metrizable?)
9. Show that if $f : X \to Y$ is measurable function, then for every $B \in \mathcal{B}(Y)$, the set $f^{-1}(B)$ is measurable.
10. Let $f : X \to \overline{\mathbb{R}}$ be a simple function written as $f = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$, where the α_i are distinct.
    (a) Show that f is measurable if each A_i is measurable.
    (b) Give an example to illustrate the necessity of the distinctness of the α_i in order to show that if f is measurable, then each A_i is measurable.
11. Let X be a measure space, and let $f : X \to [0, \infty]$ be measurable. Show that there exists a sequence of simple, measurable functions $\{f_n\}_{n=1}^{\infty}$ that converges pointwise almost everywhere to f.