1. (*) Read Kolmogorov and Fomin, Chapter 4.

2. (*) Read Sally, Chapter 5, especially Section 4.

3. Sally, Section 5.4, Exercises (*) 5.4.2, (*) 5.4.3, 5.4.5, (*) 5.4.8, 5.4.10, (*) 5.4.16, 5.4.17, 5.4.18, 5.4.19, (*) 5.4.20, (*) 5.4.21.

4. Let $O_n(\mathbb{R}) = \{ A \in GL_n(\mathbb{R}) \mid A^tA = I \}$ be the orthogonal group of $n \times n$ real matrices.
   
   (a) Show that $A \in O_n(\mathbb{R})$ if and only if $\langle Ax, Ay \rangle = \langle x, y \rangle$, $\forall x, y \in \mathbb{R}^n$.
   
   (b) Show that $A \in O_n(\mathbb{R})$ if and only if $\|Ax\| = \|x\|$, $\forall x \in \mathbb{R}^n$.
   
   (c) Show that $A \in O_n(\mathbb{R})$ if and only if the columns of $A$ form an orthonormal basis for $\mathbb{R}^n$.
   
   (d) Show that $O_n(\mathbb{R})$ is compact.

5. (Iwasawa Decomposition)
   
   Let $G = GL_n(F)$ for $F = \mathbb{R}$ or $\mathbb{C}$.
   
   Let $K = O_n(\mathbb{R})$ or $U_n(\mathbb{C})$ when $F = \mathbb{R}$ or $\mathbb{C}$, respectively.
   
   Let $A = \{ [\alpha_{ij}] \in G \mid \alpha_{ij} = 0 \text{ when } i \neq j \}$ be the diagonal matrices.
   
   Let $N = \{ [\alpha_{ij}] \in G \mid \alpha_{ii} = 1, \forall i, \text{ and } \alpha_{ij} = 0 \text{ when } i > j \}$ be the unipotent upper-triangular matrices.
   
   Show that $G = KAN$.

6. (Diagonalizable Matrices)
   
   A matrix $D = [δ_{ij}] \in M_n(F)$ is diagonal if $δ_{ij} = 0$ whenever $i \neq j$. A matrix $A \in M_n(F)$ is said to be diagonalizable if there exists $S \in GL_n(F)$ such that $D = SAS^{-1}$ is a diagonal matrix.
   
   (a) Show that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not diagonalizable as an element of $GL_2(\mathbb{R})$.
   
   (b) Show that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable as an element of $GL_2(\mathbb{C})$.
   
   (c) Show that $B = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is not diagonalizable as an element of $GL_2(F)$ for any $F$.
   
   (d) Determine whether the set of diagonalizable matrices in $GL_n(\mathbb{R})$ is open, closed, or neither.

7. Let $V = F^n$ be a vector space, and let $L : V \rightarrow V$ be a linear transformation with matrix $A \in M_n(F)$ with respect to the standard basis. A scalar $λ \in F$ is called an eigenvalue of $A$ (and of $L$) if there exists a non-zero vector $v \in V$ such that $Av = λv$. If $λ$ is an eigenvalue of $A$, then any $v \in V$ satisfying $Av = λv$ is called a corresponding eigenvector, and the collection of all such vectors $E_λ = \{ v \in V \mid Av = λv \}$ is called the eigenspace of $λ$. The characteristic polynomial of $A$ is the polynomial $p_A(λ) = \det(A - λI)$.
   
   (a) Show that $λ \in F$ is an eigenvalue of $A$ if and only if $p_A(λ) = 0$.
   
   (b) Show that if $v_1, \ldots, v_m$ are non-zero eigenvectors for $A$ with distinct eigenvalues $λ_1, \ldots, λ_m$, respectively, then the set $\{ v_1, \ldots, v_m \}$ is linearly independent.
   
   (c) Show that if $B = SAS^{-1}$ for some $S \in GL_n(F)$, then $B$ has precisely the same set of eigenvalues as $A$.
   
   (d) Show that if $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.
8. Let $F$ be a field, and let $k = F(t)$ be the field of rational functions with coefficients in $F$. Define a valuation on the non-zero elements of $k$ by $v : k \to \mathbb{Z}$ with $v(a(t)/b(t)) = \deg(b) - \deg(a)$ and an absolute value $|\cdot| : k \to \mathbb{R}$ by $|a(t)/b(t)| = e^{-v(a(t)/b(t))}$ and $|0| = 0$.

Show that $|\cdot|$ defines a non-Archimedean absolute value on $k$.

9. Show that a non-Archimedean field is totally disconnected.

10. Fix a prime $p \in \mathbb{Z}$, and consider the $p$-adic field $\mathbb{Q}_p$.

Let $x \in \mathbb{Q}_p$, and suppose that $x = \sum_{k=\nu(x)}^{\infty} a_k p^k$ is its $p$-adic expansion.

(a) Show that the $p$-adic expansion of $x$ repeats if and only if $x \in \mathbb{Q}$.

(b) If $x \in \mathbb{Q}$ with $x = \frac{a}{b}$ in lowest terms, determine the length of the repeating part of the $p$-adic expansion of $x$ in terms of $a$, $b$, and $p$. 