Math 207, Section 31: Honors Analysis I
Autumn Quarter 2009
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Homework 9, Version 3
Due: Wednesday, December 2, 2009

1. (*) Read Kolmogorov and Fomin, Chapter 4.
2. (*) Read Sally, Chapter 5, especially Section 4.
3. Sally, Section 5.4, Exercises (*) 5.4.2, (*) 5.4.3, 5.4.5, (*) 5.4.8, 5.4.10, (*) 5.4.16, 5.4.17, 5.4.18, 5.4.19, (*) 5.4.20, (*) 5.4.21.
4. Let $O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^tA = I\}$ be the orthogonal group of $n \times n$ real matrices.
   (a) Show that $A \in O_n(\mathbb{R})$ if and only if $\langle Ax, Ay\rangle = \langle x, y\rangle, \forall x, y \in \mathbb{R}^n$.
   (b) Show that $A \in O_n(\mathbb{R})$ if and only if $||Ax|| = ||x||, \forall x \in \mathbb{R}^n$.
   (c) Show that $A \in O_n(\mathbb{R})$ if and only if the columns of A form an orthonormal basis for $\mathbb{R}^n$.
   (d) Show that $O_n(\mathbb{R})$ is compact.
5. (Iwasawa Decomposition)
   Let $G = GL_n(F)$ for $F = \mathbb{R}$ or $\mathbb{C}$.
   Let $K = O_n(\mathbb{R})$ or $U_n(\mathbb{C})$ when $F = \mathbb{R}$ or $\mathbb{C}$, respectively.
   Let $A = \{[\alpha_{ij}] \in G \mid \alpha_{ij} = 0 \text{ when } i \neq j\}$ be the diagonal matrices.
   Let $N = \{[\alpha_{ij}] \in G \mid \alpha_{ii} = 1, \forall i, \text{ and } \alpha_{ij} = 0 \text{ when } i > j\}$ be the unipotent upper-triangular matrices.
   Show that $G = KAN$.
6. (Diagonalizable Matrices)
   A matrix $D = [\delta_{ij}] \in M_n(F)$ is *diagonal* if $\delta_{ij} = 0$ whenever $i \neq j$. A matrix $A \in M_n(F)$ is said to be *diagonalizable* if there exists $S \in GL_n(F)$ such that $D = SAS^{-1}$ is a diagonal matrix.
   (a) Show that $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is not diagonalizable as an element of $GL_2(\mathbb{R})$.
   (b) Show that $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is diagonalizable as an element of $GL_2(\mathbb{C})$.
   (c) Show that $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is not diagonalizable as an element of $GL_2(F)$ for any F.
   (d) Determine whether the set of diagonalizable matrices in $GL_2(\mathbb{R})$ is open, closed, or neither.
7. Let $V = F^n$ be a vector space, and let $L : V \to V$ be a linear transformation with matrix $A \in M_n(F)$ with respect to the standard basis. A scalar $\lambda \in F$ is called an *eigenvalue* of A (and of L) if there exists a non-zero vector $v \in V$ such that $Av = \lambda v$. If λ is an eigenvalue of A, then any $v \in V$ satisfying $Av = \lambda v$ is called a corresponding *eigenvector*, and the collection of all such vectors $E_\lambda = \{v \in V \mid Av = \lambda v\}$ is called the *eigenspace* of λ. The *characteristic polynomial* of A is the polynomial $p_A(\lambda) = det(A - \lambda I)$.
   (a) Show that $\lambda \in F$ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.
   (b) Show that if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are non-zero eigenvectors for A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_m$, respectively, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent.
   (c) Show that if $B = SAS^{-1}$ for some $S \in GL_n(F)$, then B has precisely the same set of eigenvalues as A.
   (d) Show that if A has n distinct eigenvalues, then A is diagonalizable.

8. Let F be a field, and let $k = F(t)$ be the field of rational functions with coefficients in F. Define a valuation on the non-zero elements of k by $v : k \to \mathbb{Z}$ with $v(a(t)/b(t)) = \deg(b) - \deg(a)$ and an absolute value $|\cdot| : k \to \mathbb{R}$ by $|a(t)/b(t)| = e^{-v(a(t)/b(t)}$ and $|0| = 0$.

   Show that $|\cdot|$ defines a non-Archimedean absolute value on k.

9. Show that a non-Archimedean field is totally disconnected.

10. Fix a prime $p \in \mathbb{Z}$, and consider the p-adic field $\mathbb{Q}_p$.

    Let $x \in \mathbb{Q}_p$, and suppose that $x = \sum_{k=val(x)}^{\infty} a_k p^k$ is its p-adic expansion.

    (a) Show that the p-adic expansion of x repeats if and only if $x \in \mathbb{Q}$.

    (b) If $x \in \mathbb{Q}$ with $x = \frac{a}{b}$ in lowest terms, determine the length of the repeating part of the p-adic expansion of x in terms of a, b, and p.