Math 208, Section 31: Honors Analysis II
Winter Quarter 2010
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Homework 9, Version 3
Due: WEDNESDAY, March 10, 2010

1. (*) Read Kolmogorov and Fomin, Chapter 8.
2. (*) Prove that if $0 \leq f \leq g$ for measurable functions $f, g : X \to [0, \infty]$, then $0 \leq \int_X f\, d\mu \leq \int_X g\, d\mu$.
3. (*) Prove that if $f : X \to [0, \infty]$ is measurable and $c \geq 0$, then $cf : X \to [0, \infty]$ is measurable, and $\int_X cf\, d\mu = c \int_X f\, d\mu$.
4. (*) Prove that if $f, g : X \to [0, \infty]$ are measurable, then $\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$.
5. Let $R \subset \mathbb{R}^n$ be a rectangle. Show that if $f : R \to \mathbb{R}$ is Riemann integrable on R, then f is Lebesgue integrable on R and the values of the two integrals coincide.
6. Prove that if $f : X \to \mathbb{C} \cup \{\infty\}$ is integrable, then $|\int_X f\, d\mu| \leq \int_X |f|\, d\mu$.
   (Hint: Choose $c \in \mathbb{C}$ such that $|c| = 1$ and $|\int_X f\, d\mu| = \int_X cf\, d\mu$.)
7. Let $E \subset X$ be a measurable set, and let $f : E \to \mathbb{C} \cup \{\infty\}$ be a measurable function. Show that $\int_E |f(x)|\, d\mu = 0$ if and only if $f(x) = 0$ almost everywhere on E.
8. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{C} \cup \{\infty\}$ be an integrable function. Show that: given $\varepsilon > 0$, there exists $\delta > 0$ such that if $A \in \mathcal{M}$ satisfies $\mu(A) < \delta$, then $\int_A |f|\, d\mu < \varepsilon$.
9. Let X be a metric space and μ a regular Borel measure on X. Given an integrable function $f : X \to \mathbb{C}$ and an $\varepsilon > 0$, show that there exists $\phi \in C_c(X; \mathbb{C})$ such that $\int_X |f - \phi|\, d\mu < \varepsilon$.

   Recall that the *support* of a function is defined to be the closure of the set $A = \{x \in X \mid f(x) \neq 0\}$ and that $C_c(X; \mathbb{C}) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and has compact support}\}$.
10. If $f : \mathbb{R}^n \to \mathbb{C}$ is integrable, show that $\int_{\mathbb{R}^n} |f(x + y) - f(x)|\, dm \longrightarrow 0$ as $y \to 0$.

Recall that if $1 \leq p < \infty$, we define $L^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f|^p\, dm < \infty\}$ (or as equivalence classes of such functions where two functions are equivalent if they differ only on a set of measure zero). If $f \in L^p(\mathbb{R}^n$, we define $||f||_p = (\int_{\mathbb{R}^n} |f|^p\, dm)^{1/p}$.

11. If $f, g \in L^1(\mathbb{R})$, their *convolution* is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by $(f * g)(x) = \int_{\mathbb{R}} f(t)g(x - t)\, dt$.
    (a) Show that $f * g \in L^1(\mathbb{R})$.
    (b) Show that convolution is commutative and associative on $L^1(\mathbb{R})$.
12. Let $f \in L^1(\mathbb{R})$, let $g \in L^p(\mathbb{R})$, and define $f * g$ by the same formula as above.
    (a) Show that $f * g \in L^p(\mathbb{R})$.
    (b) Show that for a fixed $f \in L^1(\mathbb{R})$, the map $g \mapsto f * g$ is a bounded linear operator on $L^p(\mathbb{R})$ whose norm is $||f||_1$.
13. Assume $1 \leq p < q \leq \infty$. Let m be Lebesgue measure on $\mathbb{R}$.
    (a) Find a function f that is in $L^p(\mathbb{R}, m)$ but not $L^q(\mathbb{R}, m)$.
    (b) Find a function g that is in $L^q(\mathbb{R}, m)$ but not $L^p(\mathbb{R}, m)$.

    (Hint: You might need separate examples for the case $q = \infty$.)
14. Show that $L^\infty(\mu)$ is a normed linear space with the essential supremum norm.

For the closed interval $[a,b] \subset \mathbb{R}$, we define $BV[a,b] = \{f : [a,b] \to \mathbb{R} \mid V_a^b f < \infty\}$, that is, the functions of *bounded variation* on $[a,b]$. Here, we define $V_a^b f = \sup_P \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$ where P runs over all finite partitions $a = x_0 < x_1 < \cdots < x_n = b$.

15. Show that $BV[a,b]$ is a normed linear space with norm $||f||_{BV} = |f(a)| + V_a^b f$.

16. (a) Show that if $g, h : [a,b] \to \mathbb{R}$ are non-decreasing, then $g - h \in BV[a,b]$.
    (b) Show that if $f \in BV[a,b]$, then there exist $g, h : [a,b] \to \mathbb{R}$ that are non-decreasing such that $f = g - h$.

17. (a) Show that if $f \in BV[a,b]$, then f is differentiable almost everywhere on $[a,b]$.
    (b) Show that if $f \in BV[a,b]$, then $f' \in L^1([a,b])$ and $\int_a^b f'\, dm \leq f(b) - f(a)$.

18. (a) If $f \in L^1([a,b])$, define $F(x) = \int_a^x f\, dm$. Show that $V_a^b F \leq \int_a^b |f|\, dm$ and hence $F \in BV[a,b]$.
    (b) Show that if $f_n : [a,b] \to \mathbb{R}$ is a sequence of non-decreasing functions and $f = \sum f_n$ pointwise almost everywhere, then $f' = \sum f_n'$ almost everywhere.
    (c) If $f \in L^1([a,b])$ and $F(x) = \int_a^x f\, dm$, show that $F'(x) = f(x)$ for almost all x.

19. Show that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then $f \in BV[a,b]$.

20. Show that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous and $f' = 0$ almost everywhere, then f is constant.