Math 258, Section 31: Honors Algebra II
Winter Quarter 2009
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Homework 1, Version 2 (final, with the typos in # 6 corrected)
Due: Friday, January 9, 2009

From Homework 1 onward, starred (*) problems are to be considered "moral" homework, which means that you are responsible for the material, but the problems do not need to be written up and submitted for grading.

1. (*) Read Dummit and Foote, Sections 7.1–7.3.
2. (*) Dummit and Foote, Section 7.1, #1–6 and 28.
3. (*) Give an example of a ring S and a subring R such that both S and R have identity elements but the identity element of R is not the same as that of S.
4. Dummit and Foote, Section 7.1, #7:

   The *center* of a ring R is the set $\{z \in R \mid zr = rz \text{ for all } r \in R\}$. Prove that the center of a ring is a subring that contains the identity. Prove that the center of a division ring is a field.
5. Dummit and Foote, Section 7.1, #12:

   Prove that any subring of a field which contains the identity is an integral domain.
6. Dummit and Foote, Section 7.1, a blend of #13 and 14:

   An element $x \in R$ is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$. For the following, suppose x is a nilpotent element of a commutative ring with identity R.

   (a) Prove that x is either zero or a zero divisor.
   (b) Prove that rx is nilpotent for all $r \in R$.
   (c) Prove that $1 + x$ is a unit in R.
   (d) Deduce that the sum of a nilpotent element and a unit is a unit.
7. Dummit and Foote, Section 7.1, #17:

   Let R and S be rings. Prove that the direct product $R \times S$ is a ring under componentwise addition and multiplication. Prove that $R \times S$ is commutative if and only if R and S are commutative. Prove that $R \times S$ has an identity if and only if both R and S have identities.
8. A rephrasing of Dummit and Foote, Section 7.1, # 29:

   Prove that the endomorphisms from an Abelian group to itself form a ring with identity, where the addition is pointwise addition and the multiplication is by function composition. Prove that the units of this ring are the group automorphisms of the Abelian group.
9. Prove that for D an odd integer that is not a square, the set
   $$\mathbb{Z}[(1+\sqrt{D})/2] = \{a + b(1+\sqrt{D})/2 \mid a, b \in \mathbb{Z}\}$$
   is a ring if and only if $D \equiv 1 \pmod 4$.
10. Let R be a commutative ring with identity. Suppose f and g are polynomials in $R[x]$. Prove that for any $a \in R$, we have:
    $$(f+g)(a) = f(a) + g(a)$$
    $$(fg)(a) = f(a)g(a)$$
    where $f(a)$ denotes the polynomial f evaluated setting $x = a$.