Math 258, Section 31: Honors Algebra II
Winter Quarter 2009
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Homework 2, Version 3 (the rings in 9(d), 11, 12, and 14 should all have 1's)
Due: Friday, January 16, 2009

Reminder: starred (*) problems are to be considered "moral" homework.

1. (*) Read Dummit and Foote, Sections 7.3–7.5.
2. (*) Dummit and Foote, Section 7.3, #1–12 and 28.
3. Dummit and Foote, Section 7.3, #12, 13, and 14:
   Let $D \in \mathbb{Z}$ be an integer that is not a perfect square, and let
   $$S = \{ \left[\begin{smallmatrix} a & b \\ Db & a \end{smallmatrix} \right] \mid a, b \in \mathbb{Z} \}.$$
   (a) Prove that S is a subring of $M_2(\mathbb{Z})$.
   (b) Prove that the map $\varphi : \mathbb{Z}[\sqrt{D}] \to S$ given by:
   $$\varphi(a + b\sqrt{D}) = \left[\begin{smallmatrix} a & b \\ Db & a \end{smallmatrix} \right]$$
   is a ring homomorphism.
   (c) If $D \equiv 1 \pmod 4$ is square-free, prove that the set
   $$S = \{ \left[\begin{smallmatrix} a & b \\ (D-1)b/4 & a+b \end{smallmatrix} \right] \mid a, b \in \mathbb{Z} \}$$
   is a subring of $M_2(\mathbb{Z})$ that is isomorphic to the quadratic integer ring $\mathcal{O}$.
   (d) Prove that the ring $M_2(\mathbb{R})$ contains a subring that is isomorphic to $\mathbb{C}$.
   (e) Prove that the ring $M_4(\mathbb{R})$ contains a subring that is isomorphic to $\mathbb{H}$.
4. Dummit and Foote, Section 7.3, #18 and 19:
   Let R be a ring.
   (a) If $\{I_\alpha \mid \alpha \in A\}$ is a family of ideals in R, then $\bigcap_{\alpha \in A} I_\alpha$ is an ideal in R.
   (b) If $I_1 \subset I_2 \subset I_3 \subset \cdots$ is a (nested, countable) collection of ideals in R, then $\bigcup_{n \in \mathbb{N}} I_n$ is an ideal in R.
5. Dummit and Foote, Section 7.3, #22:
   Let a be an element of the ring R.
   (a) Prove that $\{x \in R \mid ax = 0\}$ is a right ideal and that $\{y \in R \mid ya = 0\}$ is a left ideal. (These are called the *left-* and *right-annihilators* of a in R.)
   (b) Prove that if L is a left ideal in R, then $\{x \in R \mid xa = 0, \forall a \in L\}$ is a two-sided ideal. (This is called the *annihilator* of L in R.)
6. (*) Prove the First and Fourth Isomorphism Theorems for rings.
7. Prove the Third Isomorphism Theorem for rings.

8. A modification/combination of Dummit and Foote, Section 7.3, #29 and 30:

   Let R be a commutative ring. We define $\mathfrak{N}(R)$ to be the set of nilpotent elements in R.

   (a) Show that $\mathfrak{N}(R)$ is an ideal in R.
   (b) Find $\mathfrak{N}(R)$ for:
      i. $R = \mathbb{Z}/n\mathbb{Z}$ for all values of $n \geq 2$
      ii. $R = M_2(\mathbb{R})$
      iii. $R = M_2(\mathbb{Z}/4\mathbb{Z})$
   (c) Prove that 0 is the only nilpotent element in $R/\mathfrak{N}(R)$.

9. Dummit and Foote, Section 7.3, #34:

   Let I and J be ideals in the ring R.

   (a) Prove that $I + J$ is the smallest ideal of R containing both I and J.
   (b) Prove that IJ is an ideal contained in $I \cap J$.
   (c) Give an example where $IJ \neq I \cap J$.
   (d) Prove that if R is a commutative ring with 1 and $I + J = R$, then $IJ = I \cap J$.

10. (*) Dummit and Foote, Section 7.4, #4–8 and 14–17.

11. Dummit and Foote, Section 7.4, #10:

    Assume R is a commutative ring with $1 \neq 0$. Prove that if P is a prime ideal of R and P contains no zero divisors, then R is an integral domain.

12. Dummit and Foote, Section 7.4, a modification of #19:

    Prove that if R is a finite commutative ring with $1 \neq 0$, then every prime ideal of R is maximal.

13. Dummit and Foote, Section 7.4, #33:

    Let R be the ring of all continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$. For each $c \in [0, 1]$, let $M_c = \{f \in R \mid f(c) = 0\}$.

    (a) Prove that if M is any maximal ideal in R, then there exists $c \in [0, 1]$ such that $M = M_c$.
    (b) Prove that if b and c are distinct points in $[0, 1]$, then $M_b \neq M_c$.
    (c) Prove that M_c is not equal to the principal ideal generated by $(x - c)$.
    (d) Prove that M_c is not finitely generated.

14. Dummit and Foote, Section 7.4, #37:

    A commutative ring with 1 is called a *local ring* if it has a unique maximal ideal.

    (a) (*) Give an example of a local ring.
    (b) Prove that if R is a local ring with unique maximal ideal M, then every element of $R \setminus M$ is a unit.
    (c) Prove that if R is a commutative ring with identity in which the set of non-units forms an ideal M, then R is a local ring with unique maximal ideal M.