Math 258, Section 31: Honors Algebra II
Winter Quarter 2009
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Homework 5, Final Version
Due: MONDAY, February 16, 2009

1. (*) Read Dummit and Foote, Sections 9.1–9.5.
2. (*) Dummit and Foote, Section 9.3, #2, 3.
3. (*) Dummit and Foote, Section 9.4, #1–4, 11–13, 18.
4. Find all irreducible monic polynomials of degree 4 or less in $k[x]$ for:
   (a) $k = \mathbb{Z}/2\mathbb{Z}$
   (b) $k = \mathbb{Z}/3\mathbb{Z}$
5. A combination of Dummit and Foote, Section 9.2, #2 and 3, and Section 9.4, #6:
   (a) Let k be a finite field of order q, and let $p(x) \in k[x]$ be a polynomial of degree $n \geq 1$. Show that $k[x]/(p(x))$ is a ring of order q^n that is a field iff $p(x)$ is irreducible.
   (b) Construct fields of order 7^2, 7^3, and 7^4.
6. Dummit and Foote, Section 9.4, #14.
   Factor the polynomials $x^8 - 1$ and $x^6 - 1$ into irreducibles when considered as elements of $R[x]$ for:
   (a) $R = \mathbb{Z}$
   (b) $R = \mathbb{Z}/2\mathbb{Z}$
   (c) $R = \mathbb{Z}/3\mathbb{Z}$
7. Dummit and Foote, Section 9.4, #20.
   Let $f(x) = x \in \mathbb{Z}/6\mathbb{Z}$.
   (a) Show that $f(x) = (3x + 4)(4x + 3)$ and hence is not irreducible.
   (b) Show that the reduction of $f(x)$ modulo both of the non-trivial ideals (2) and (3) is an irreducible polynomial and that, hence, the condition in Proposition 12 that R be an integral domain is necessary.
   (c) Show that in any factorization $f(x) = g(x)h(x)$ in $\mathbb{Z}/6\mathbb{Z}$, the reduction of both $g(x)$ or $h(x)$ modulo (2) is either x or 1, and that the result is similar for reduction modulo (3). Determine all of the factorizations of $f(x)$ in $\mathbb{Z}/6\mathbb{Z}$.
   (d) Show that $f(x) = x \in \mathbb{Z}/30\mathbb{Z}$ has the factorization $f(x) = (10x + 21)(15x + 16)(6x + 25)$. Prove that the product of any two of these factors is again of the same degree. Prove that the reduction of $f(x)$ modulo any prime in $\mathbb{Z}/30\mathbb{Z}$ is an irreducible polynomial. Determine all of the factorizations of $f(x) = x$ in $\mathbb{Z}/30\mathbb{Z}$.
8. (a) Suppose R is an integral domain. Prove that R is infinite if and only if for every nonzero polynomial $f(x) \in R[x]$, there exists $a \in R$ such that $f(a) \neq 0$.
   (b) Suppose R is an infinite integral domain and $n \geq 1$. Prove that for any nonzero polynomial $f(x_1, x_2, \ldots, x_n) \in R[x_1, x_2, \ldots, x_n]$, there exist $a_1, a_2, \ldots, a_n \in R$ such that $f(a_1, a_2, \ldots, a_n) \neq 0$.
   (c) Suppose S is a non-empty set and $R = \mathcal{P}(\mathcal{S})$ denotes the ring of all functions from S to the ring $\mathbb{Z}/2\mathbb{Z}$, with pointwise addition and multiplication. Find a nonzero polynomial $f(x) \in R[x]$ such that $f(a) = 0$ for all $a \in R$.

9. Prove that the polynomial $x^2 + 1$ has uncountably many roots in the skew field $\mathbb{H}$ of Hamiltonian quaternions.

10. Dummit and Foote, Section 9.6, #1:

    Let F be a field. Suppose I is an ideal in $F[x_1, \ldots, x_n]$ generated by a (possibly infinite) set $\mathcal{S}$ of polynomials. Prove that a finite subset of the polynomials in $\mathcal{S}$ suffice to generate I.