Math 258, Section 31: Honors Algebra II
Winter Quarter 2009
John Boller
Homework 6, Final Version
Due: MONDAY, February 23, 2009

1. (*) Read Dummit and Foote, Sections 11.1–11.4.
2. (*) Dummit and Foote, Section 11.1, # 1-7.
3. (*) Show that if $\{v_1, \dots, v_n\}$ is a basis for V, then the representation of a given $v \in V$ as a linear combination of the basis elements is unique.
4. (*) If V and W are vector spaces and $L : V \to W$ is a linear map, show:
   (a) $0 \cdot v = 0, \forall v \in V$
   (b) $(-1) \cdot v = -v, \forall v \in V$
   (c) $L(0) = 0$
   (d) $L(-v) = -L(v), \forall v \in V$
   (e) $Ker(L)$ is a subspace of V
   (f) $Im(L)$ is a subspace of W
5. Let $F = \mathbb{F}_q$ be the finite field with q elements.
   (a) Find, with proof, the number of 1-dimensional subspaces of F^n $(n \geq 1)$.
   (b) Find, with proof, the number of 2-dimensional subspaces of F^n $(n \geq 2)$.
   (c) Generalize to find, with proof, the number of m-dimensional subspaces of F^n $(n \geq m)$.
6. (a) Prove that every linear map $L : F^2 \to F^2$ has the form

   $$L(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$$

   for some constants $\alpha, \beta, \gamma, \delta \in F$.
   (b) Prove that such a linear map is invertible iff $\alpha\delta - \beta\gamma \neq 0$.
7. (a) Prove that if $dim(V) = dim(W) < \infty$, then a linear map $L : V \to W$ is injective iff it is surjective.
   (b) Show by example that the result if false if V and W are both infinite-dimensional.
8. (*) Let V be a vector space over F and W a subspace of V.
   We define the *quotient space* $V/W = \{v + W \mid v \in V\}$ with operations:

   $$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

   $$\alpha(v + W) = (\alpha v) + W$$

   (a) Show that these operations are well-defined and that, with them, V/W is a vector space over F.
   (b) Read Dummit and Foote's Theorem 7 in Section 11.1.
   (c) Prove the Vector Space Isomorphism Theorems:
      i. (First) If $L : V \to W$ is linear, then $Im(L) \cong V/Ker(L)$.
      ii. (Second) If V and W are subspaces of some vector space U, then $(V + W)/W \cong V/(V \cap W)$.
      iii. (Third) If U is a subspace of W, which is in turn a subspace of V, then $(V/U)(W/U) \cong V/W$.

iv. (Fourth) If W is a subspace of V, then there is a bijection between the subspaces of V that contain W and the subspaces of V/W given by $U \leftrightarrow U/W$.

9. Suppose that $L : V \to W$ is a linear bijection. Prove that $L^{-1} : W \to V$ is linear.

10. Prove that any vector space has a basis.

11. Given two vector spaces V and W over the same field F, we define their *direct sum* to be

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}$$

with vector space operations given by:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

$$\alpha(v, w) = (\alpha v, \alpha w)$$

$\forall \alpha \in F, \forall v, v_1, v_2 \in V, \forall w, w_1, w_2 \in W$.

   (a) (*) Prove that $V \oplus W$ is a vector space with these operations.
   (b) (*) Prove that $U \oplus (V \oplus W) \cong (U \oplus V) \oplus W$.
   (c) If V and W are subspaces of a common finite-dimensional vector space U, we define $V + W = \{v+w \mid v \in V, w \in W\}$. It is easy to see that $V+W$ is a subspace of U. Prove that $V+W \cong V \oplus W$ iff $V \cap W = \{0\}$.

12. A linear transformation $P : V \to V$ is called a *projection* if $P^2 = P$, where $P^2 = P \circ P$. For a projection $P : V \to V$, let $E_0 = Ker(P)$ and $E_1 = \{v \in V \mid P(v) = v\}$. Prove that $V \cong E_0 \oplus E_1$.

13. (*) (Sorry, out of order, but I had a request not to put new problems at the front of the list.)
   (a) Show that $f(x) = x^3 + 3x^2 - 8$ is irreducible in $\mathbb{Q}[x]$.
   (b) Show that $f(x) = x^4 - 22x^2 + 1$ is irreducible in $\mathbb{Q}[x]$.
   (c) Decide whether or not $f(x) = x^6 - 12$ is irreducible in $\mathbb{Q}[x]$.
   (d) Decide whether or not $f(x) = 2x^7 - 25x^3 + 10x - 30$ is irreducible in $\mathbb{Q}[x]$.

14. Show that $f(x) = x^4 - 2x^2 + 9$ is irreducible in $\mathbb{Q}[x]$ but that there is no $c \in \mathbb{Z}$ such that $f(x + c)$ satisfies Eisenstein's criterion.

15. Let V and W be finite-dimensional vector spaces over F. Prove that

$$dim(Hom_F(V, W)) = dim(V) \cdot dim(W).$$

16. (*) Prove that $M_n(F)$ is a ring and that $GL_n(F) = \{A \in M_n(F) \mid A \text{ is invertible}\}$ is a group, which is non-commmutative for $n \geq 2$.

17. (*) Let $V = \mathbb{R}^2$ be two-dimensional Euclidean space, with its usual x- and y- coordinate axes. Consider the linear transformation $L_\alpha : V \longrightarrow V$ that performs a reflection about the line $y = \alpha x$.
   (a) Write the matrix for L_α with respect to the basis $\mathfrak{B} = \{e_1, e_2\}$. (Hint: Use elementary geometry to compute $L_\alpha(e_1)$ and $L_\alpha(e_2)$.)
   (b) Calculate the matrix for $L_\beta \circ L_\alpha$ (with respect to $\mathfrak{B}$) in two ways: by multiplying the matrices for L_β and L_α, and by determining the matrix for the resulting composed linear transformation directly.
   (c) Show that the composed linear transformation $L_\beta \circ L_\alpha$ is a rotation. By what angle are vectors in $\mathbb{R}^2$ rotated under this transformation?

18. (*) If $A = [a_{ij}]$ is an $n \times m$ matrix then we define its *transpose* $A^t = [a_{ji}]$ to be the $m \times n$ matrix whose rows are the columns of A. That is,

$$\text{if } A = \begin{bmatrix} a_{11} & \cdot \; \cdot \; \cdot & a_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdot \; \cdot \; \cdot & a_{nm} \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_{11} & \cdot \; \cdot \; \cdot & a_{n1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{1m} & \cdot \; \cdot \; \cdot & a_{mn} \end{bmatrix}.$$

(a) Prove that if A and B are $n \times m$, then $(A + B)^t = A^t + B^t$.

(b) Prove that if A is $n \times m$ and B is $m \times k$, then $(AB)^t = B^t A^t$.

(c) Prove that if A is $n \times n$, then $det(A^t) = det(A)$.

19. (*) Dummit and Foote, Section 11.2, #1–7, 14–37.

20. (*) Let $V = \mathbb{R}^n$, and let $\mathbf{u}, \mathbf{v} \in V$. If $A : V \longrightarrow V$ is (the matrix for) a linear transformation, then define the following bilinear form:

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}$$

(a) Show that f_A is indeed a bilinear form.

(b) Give a necessary and sufficient conditions on the matrix A that makes f_A alternating.

(Hint #1: You might consider the $n = 2$ case first to get a feel for this bilinear form. Hint #2: Your answers just might involve the transpose!)

21. (*) Show that not every skew-symmetric multilinear form $f : V^n \longrightarrow F$ is alternating by constructing an example. (Note that the only cases where this can happen are over fields F wherein $1 + 1 = 0$.)

22. Let $dim(V) = n$, and let $f : V^k \longrightarrow F$ be a non-trivial alternating k-linear form with $k < n$. Show by example that it is possible to have a set of k linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in V such that $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$. (Make sure that $k \geq 2$ so that f can be alternating!)

23. Let V be a vector space over F. Consider the set of k-linear forms $f : V^k \longrightarrow F$. For any two such forms f_1 and f_2 and any scalar $c \in F$, we define:

$$(f_1 + f_2)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = f_1(\mathbf{v}_1, \ldots, \mathbf{v}_k) + f_2(\mathbf{v}_1, \ldots, \mathbf{v}_k), \qquad \forall \mathbf{v}_1, \ldots, \mathbf{v}_k \in V$$

$$(cf_1)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = c \cdot f_1(\mathbf{v}_1, \ldots, \mathbf{v}_k), \qquad \forall \mathbf{v}_1, \ldots, \mathbf{v}_k \in V$$

(a) (*) Convince yourself that the collection of k-linear forms on V form a vector space over F with addition and scalar multiplication defined as above.

(b) (*) Let $V = \mathbb{R}^2$. Show that the form $f : V^2 \longrightarrow \mathbb{R}$ defined by $f((a, b), (c, d)) = ad - bc$ is bilinear and alternating.

(c) Now let $V = \mathbb{R}^3$. Construct two *linearly independent* alternating bilinear forms $f : V^2 \longrightarrow \mathbb{R}$.

(d) Determine the dimensions of the spaces of alternating bilinear forms on $V = \mathbb{R}^2$ and $V = \mathbb{R}^3$.