Math 258, Section 31: Honors Algebra II
Winter Quarter 2009
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Homework 7, Final Version
Due: Friday, March 6, 2009

1. (*) Read Dummit and Foote, Sections 11.1–11.4 and Sections 10.1–10.3.

2. (*) Dummit and Foote, Section 11.4, # 1-6.

3. Consider the linear map $L : (\mathbb{Z}/7\mathbb{Z})^3 \longrightarrow (\mathbb{Z}/7\mathbb{Z})^3$ given by

$$L(x, y, z) = (x + y + z, 2x + 3y + 4z, 3x + 4y + 6z),$$

   and let A be the matrix of L with respect to the standard basis.

   (a) Find the inverse of A.

   (b) Determine the eigenvalues of A.

   (c) Determine the corresponding eigenspaces of A.

4. Suppose λ is an eigenvalue for the linear transformation $A : V \longrightarrow V$.

   (a) Show that λ^n is an eigenvalue for A^n for any $n \in \mathbb{N}$.

   (b) If A is invertible, show that λ^{-1} is an eigenvalue for A^{-1}, and that, therefore, λ^n is an eigenvalue for A^n for any $n \in \mathbb{Z}$. (By the way, (*) what happens if $\lambda = 0$?)

5. Find examples of *invertible* linear transformations $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that:

   (a) A has no real eigenvalues.

   (b) A has only one eigenvalue λ, but $dim(V_\lambda) < 4$.

   (c) $\mathbf{e}_1 = (1, 0, 0, 0)$ and $\mathbf{v} = (1, 1, 1, 1)$ are both eigenvectors but have distinct eigenvalues.

6. Let F be a field, $M_n(F)$ be the matrix ring over F, and $A, B \in M_n(F)$ be two matrices.

   (a) Prove that the trace of AB equals the trace of BA.

   (b) Prove that $p_{AB}(\lambda) = p_{BA}(\lambda)$.

   (c) Prove that if either A or B is invertible, then AB and BA are similar matrices.

   (d) Give an example where neither A nor B is invertible, and AB is not similar to BA. (Hint: You can restrict attention to $n = 2$, and it suffices to construct an example where $AB = 0$ but $BA \neq 0$.)

7. Let $A : V \to V$ for some finite-dimensional vector space V. Prove that the geometric multiplicity of an eigenvalue of A is less than or equal to the algebraic multiplicity of the eigenvalue.

8. Suppose K is a field and R is a ring containing K in its center (and with the same 1) such that R is n-dimensional as a K-vector space.

   (a) Construct an injective ring homomorphism $\varphi : R \to M_n(K)$, in terms of a choice of basis for R as a K-vector space.

   (b) Write down φ explicitly when $K = \mathbb{R}$ and $R = \mathbb{C}$, with basis $\{1, i\}$. Prove also that for $z \in \mathbb{C}$, the determinant of $\varphi(z)$ equals the square of the modulus of z.

   (c) Write down φ explicitly when $K = \mathbb{R}$ and $R = \mathbb{R}[x]/(x^2)$, with basis $\{1, x\}$.

   (d) When $K = \mathbb{Q}$ and $R = \mathbb{Q}[\sqrt{D}]$ with basis $\{1, \sqrt{D}\}$ (where D is a square-free integer that is neither 0 nor 1), prove that the field norm of $x \in R$ is the determinant of $\varphi(x)$.

9. (*) Dummit and Foote, Section 10.1, #1–7.

10. Dummit and Foote, Section 10.1, #8:

    An element m of the R-module M is called a *torsion element* if $rm = 0$ for some non-zero $r \in R$. The set of torsion elements is denoted

    $$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

    (a) Prove that if R is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion submodule*).

    (b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule.

    (c) If R has zero divisors, show that every non-zero R-module has non-zero torsion elements.

11. Dummit and Foote, a combination of #9, 10, 12:

    If N is a submodule of M, the *annihilator of N in R* is $\{r \in R \mid rn = 0, \forall n \in N\}$.
    If I is a right-ideal of R, the *annihilator of I in M* is $\{m \in M \mid am = 0, \forall a \in I\}$.

    (a) Prove that the annihilator of N in R is a two-sided ideal in R.

    (b) Prove that the annihilator of I in M is a submodule of M.

    (c) If N is a submodule of M and I is its annihilator in R, prove that the annihilator of I in M contains N, and give an example where the annihilator of I in M does not equal N.

    (d) If I is a right ideal in R and N is its annihilator in M, prove that the annihilator of N in R contains I, and give an example where the annihilator of N in R does not equal I.

12. (*) Dummit and Foote, Section 10.2, #1–5, 7–8:

13. Dummit and Foote, Section 10.2, #6:

    Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

14. Dummit and Foote, Section 10.2, #9:

    Let R be a commutative ring with 1. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

15. (*) Show that if V is a vector space over F and $T : V \to V$ is a linear map, then V is an $F[x]$-module with the multiplication $p(x)v = p(T)v$, for any $p(x) \in F[x]$.

16. (*) Let R be a ring with 1, and let M be a left R-module. For any subset $A \subset M$, show that $RA = \{r_1a_1 + \cdots + r_na_n \mid a_i \in A, r_i \in R, 1 \leq i \leq n, n \in \mathbb{N}\}$ is a submodule of M.

17. (*) Show that $R[x]$ is finitely generated as an $R[x]$ module but is not finitely generated as an R-module.