Math 258, Section 31: Honors Algebra II
Winter Quarter 2009
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Homework 8, Final Version
Due: WEDNESDAY, March 11, 2009

1. (*) Read Dummit and Foote, Sections 10.1–10.4.
2. (*) Dummit and Foote, Section 10.3, # 1-6.
3. A combination of Dummit and Foote, Section 10.3, #9–11.

   If R is a ring with 1 and M is an R-module, we define M to be *irreducible* if $M \neq 0$ and M has no non-trivial R-submodules.

   (a) Show that if M is irreducible, then M is cyclic, and any non-zero element is a generator.

   (b) Classify the irreducible $\mathbb{Z}$-modules.

   (c) If R is commutative, show that an R-module M is irreducible if and only if M is isomorphic to R/I for some maximal ideal $I \subset R$.

   (d) Show that if M_1 and M_2 are irreducible R-modules, then any non-zero homomorphism $\varphi : M_1 \to M_2$ is an isomorphism.

   (e) Show that if M is irreducible, then $\text{End}_R(M)$ is a division ring.
4. Show that a direct sum of free R-modules is free.
5. Show that no finite abelian group is free when considered as a $\mathbb{Z}$-module.
6. (*) Dummit and Foote, Section 10.4, #1–3, 5–6, 11–13.
7. Dummit and Foote, Section 10.4, #4:

   Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ as left $\mathbb{Q}$-modules.
8. Adapted from Dummit and Foote, Section 10.4, #18–20:

   Let R be an integral domain, and for an ideal $I \subset R$, consider the R-module $I \otimes_R I$.

   (a) Show that if I is principal, then $I \otimes_R I$ has no torsion elements.

   (b) When $R = \mathbb{Z}[x]$ and $I = (2, x)$, show that $2 \otimes x - x \otimes 2$ is a torsion element of $I \otimes_R I$.

   (c) When $R = \mathbb{Z}[x]$ and $I = (2, x)$, show that the submodule W generated by $2 \otimes x - x \otimes 2$ in $I \otimes_R I$ is isomorphic to R/I.

   (d) (*) When $R = \mathbb{Z}[x]$ and $I = (2, x)$, show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor.