Math 259, Section 33: Honors Algebra III
Spring Quarter 2009
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Homework 2, Final Version
Due: Friday, April 10, 2009

1. (*) Read Dummit and Foote, Sections 12.1–12.3.

2. Dummit and Foote, Section 12.1, #7:

   Let R be a ring. Prove that if $A_1, \dots, A_m$ are R-modules and B_i is a submodule of A_i for each $i = 1, \dots, m$, then

   $$(A_1 \oplus \cdots \oplus A_m)/(B_1 \oplus \cdots \oplus B_m) \cong (A_1/B_1) \oplus \cdots \oplus (A_m/B_m).$$

3. Find the invariant factors and elementary divisors of the following modules:

   (a) $M = (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/36\mathbb{Z}) \oplus (\mathbb{Z}/100\mathbb{Z})$ considered as a $\mathbb{Z}$-module.

   (b) $M = \mathbb{Q}[x]/((x^2+1)) \oplus \mathbb{Q}[x]/((x^4-1)) \oplus \mathbb{Q}[x]/((x^8-1)) \oplus \mathbb{Q}[x]/((x^4+1)^2)$ considered as a $\mathbb{Q}[x]$-module.

4. (*) Dummit and Foote, Section 12.2, #1–3 and 5–9.

5. Dummit and Foote, Section 12.2, # 4:

   Prove that two 3×3 matrices over a field F are similar if and only if they have the same characteristic polynomials and the same minimal polynomials. Give an explicit counter-example to this statement for 4×4 matrices.

6. Suppose K is a field and R is a ring with 1 containing K in its center, such that R is finite-dimensional as a K-vector space.

   (a) Prove that left multiplication by any element a of R is a K-linear map l_a from R to itself.

   (b) Prove that $a \in R$ satisfies a monic polynomial $p(x) \in K[x]$ if and only if the linear map l_a satisfies the same polynomial p (i.e., $p(l_a)$ is the zero linear map).

   (c) In the case where $K = \mathbb{R}$ and $R = \mathbb{C}$, write the explicit matrix for l_a where $a = u + iv$, in terms of the basis $\{1, i\}$. Determine the minimal polynomial of l_a.

7. Suppose K is a field and R is a ring with 1 containing K in its center, such that R is m-dimensional as a K-vector space. Suppose M is a free R-module of rank n. Prove that M is mn-dimensional as a K-vector space, by constructing a basis of size mn for M over K in terms of a basis for M over R and a basis for R over K.

8. Adapted from Dummit and Foote, Section 12.2, #10–11 and Example (4) from page 486:

   (a) (*) Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $m(x) = (x+1)^2(x-1)$.

   (b) Write out the Rational Canonical Form for representatives of matrices from all similarity classes of 6×6 matrices over $\mathbb{Q}$ with characteristic polynomial $m(x) = (x^4-1)(x^2-1)$.

   (c) Write out the Rational Canonical Form for representatives of matrices from all similarity classes of 6×6 matrices over $\mathbb{C}$ with characteristic polynomial $m(x) = (x^4-1)(x^2-1)$.