Math 259, Section 33: Honors Algebra III
Spring Quarter 2009
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Homework 3, Final Version
Due: Friday, April 17, 2009

1. (*) Read Dummit and Foote, Sections 12.3 and 13.1–13.2.

2. (*) Dummit and Foote, Section 12.3, #1, 4–10, 12–20.

3. Dummit and Foote, Section 12.3, #11:

   Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}$$

   (a) Verify that the characteristic polynomial of A is a product of linear factors over $\mathbb{Q}$.
   (b) Determine the rational canonical form of A.
   (c) Determine the Jordan canonical form of A.

4. Dummit and Foote, Section 12.3, #21–22:

   (a) Show that if $A^2 = A$, then A is similar to a diagonal matrix that has only 0's and 1's along the diagonal.
   (b) Prove that an $n \times n$ matrix A over $\mathbb{C}$ that satisfies $A^3 = A$ can always be diagonalized. Is this same statement true over all fields? Explain.

5. An agglomeration of Dummit and Foote, Section 12.3, #31–34:

   Recall that a matrix $A \in M_n(F)$ is said to be *nilpotent* if there is some $k \in \mathbb{N}$ such that $A^k = 0$.

   (a) Show that any nilpotent matrix is similar to a block diagonal matrix whose blocks have 0's on their diagonals and 1's on the first superdiagonals.
   (b) Show that if $A \in M_n(F)$ is nilpotent, then $A^n = 0$.
   (c) Show that if A is strictly upper-triangular, then A is nilpotent.
   (d) Prove that the trace of any nilpotent matrix is 0.

6. (a) Suppose $p(x) \in K[x]$ is a polynomial of degree n. Prove that if p has at least $n-1$ distinct roots in K, then p splits completely into linear factors over K.
   (b) Suppose p is an irreducible polynomial of degree two over a field K. Prove that p splits completely over the field $K[x]/(p(x))$.
   (c) Prove that for any natural number $n \geq 3$, we can find an irreducible polynomial $p(x) \in \mathbb{Q}[x]$ of degree n such that p does not split completely over the field $\mathbb{Q}[x]/(p(x))$.

7. The *automorphism group* of a field is defined as the group of ring isomorphisms from the field to itself. A *prime field* is a field that does not contain any proper subfield.

   (a) Prove that the only prime fields are fields of prime order and the field of rational numbers. Prove that every field contains exactly one prime subfield, and any automorphism of a field fixes every element of its prime subfield. (In particular, this shows that the automorphism groups of prime fields are trivial).

(b) Prove that the automorphism group of the field of real numbers is trivial.

(c) Prove that the automorphism group of the field $\mathbb{Q}(2^{1/3})$ is trivial.

(d) Let K be a field, and consider the field $K(t)$: the field of rational functions in one variable. Let G be the automorphism group of this field. For any $A \in GL(2, K)$, define μ_A as the following map from $K(t)$ to $K(t)$:

$$\text{For } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mu_A(f(t)) = f\left(\frac{at+b}{ct+d}\right)$$

Prove that μ_A is an automorphism of $K(t)$, and the map $A \mapsto \mu_A$ is a homomorphism of groups from $GL(2, K)$ to G. Find the kernel of this homomorphism.

8. (*) Dummit and Foote, Section 13.1, #1–8.

9. Let $F = \mathbb{F}_3$. Consider the two polynomials $p(x) = x^2 + 1$ and $q(x) = x^2 + 2x + 2$ in $F[x]$. Let $K_p = F[x]/(p(x))$ and $K_q = F[x]/(q(x))$.

   (a) (*) Show that p and q are the only two monic irreducible polynomials in $F[x]$ of degree 2.

   (b) Write down the multiplication table for $K_p = \{0, 1, 2, \theta, \theta + 1, \theta + 2, 2\theta, 2\theta + 1, 2\theta + 2\}$ where $\pi_p : F[x] \to K_p$ is the natural projection and $\theta = \pi_p(x)$.

   (c) Factor $p(x)$ over K_p.

   (d) Write down the multiplication table for $K_q = \{0, 1, 2, \eta, \eta + 1, \eta + 2, 2\eta, 2\eta + 1, 2\eta + 2\}$ where $\pi_q : F[x] \to K_q$ is the natural projection and $\eta = \pi_q(x)$.

   (e) Factor $q(x)$ completely over K_q.

   (f) Show directly that $K_p \cong K_q$.

10. (*) Dummit and Foote, Section 13.2, #1–9, 11–13.

11. Dummit and Foote, Section 13.2, #10:

    Determine the degree of the extension $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$ over $\mathbb{Q}$.

12. Dummit and Foote, Section 13.2, #14:

    Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.