Math 259, Section 33: Honors Algebra III
Spring Quarter 2009
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Homework 5
Due: Friday, May 8, 2009

1. (*) Read Dummit and Foote, Chapter 13.5–13.6 and 14.1–14.2.
2. (*) Dummit and Foote, Section 13.6, #1–5.
3. Dummit and Foote, Section 13.6, #6:

   Show that if $n > 1$ is odd, then $\Phi_{2n}(x) = \Phi_n(-x)$.
4. Dummit and Foote, Section 13.6, #9:

   Show that if $\alpha \in F$ for some field F of characteristic p, then $A = \left[\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right]$ satisfies $A^p = I$ and cannot be diagonalized unless $\alpha = 0$.
5. A concatenation of Dummit and Foote, Section 13.6, #10–12:

   Let $F = \mathbb{F}_{p^n}$, and consider the Frobenius map $\varphi : F \to F$ given by $\varphi(x) = x^p$.

   (a) Show that φ is an isomorphism.
   (b) Show that φ^n is the identity and that no lower power of φ is the identity.
   (c) Find the rational canonical form for φ acting on $F = \mathbb{F}_{p^n}$ considered as an $\mathbb{F}_p$-vector space.
   (d) Find the Jordan canonical form (over a field containing all necessary eigenvalues) for φ acting on $F = \mathbb{F}_{p^n}$ considered as an $\mathbb{F}_p$-vector space.
6. Prove a special case of Dirichlet's Theorem on primes in arithmetic progressions by doing Dummit and Foote, Section 13.6, # 14–17.
7. (*) Dummit and Foote, Section 14.1, # 1–8.
8. Adapted from Dummit and Foote, Section 14.1, # 9:

   Let k be a field, and consider $k(t)$. Show that the map $t \mapsto t + 1$ is an automorphism of $k(t)$, and determine its fixed field.
9. Dummit and Foote, Section 14.1, #10:

   Let K be an extension of F, and let $\varphi : K \to K'$ be an isomorphism of fields which maps F to the subfield F' in K'. Prove that $\sigma \mapsto \varphi\sigma\varphi^{-1}$ defines a group isomorphism $\mathrm{Aut}(K/F) \to \mathrm{Aut}(K'/F')$.
10. Give an example of a field K, a field extension L of K, and a field automorphism σ of K that cannot be extended to any field automorphism of L.
11. Give an example of a field K, a field extension L of K, and a field automorphism σ of L such that $\sigma(K) \neq K$.