Math 259, Section 33: Honors Algebra III
Spring Quarter 2009
John Boller
Homework 6, Final Version
Due: Friday, May 15, 2009

1. (*) Read Dummit and Foote, Sections 14.1–14.3.
2. (*) Dummit and Foote, Section 14.2, #1–14.
3. Suppose $K$ is a field, $L$ is a finite extension of $K$, and $M$ and $M'$ are subfields of $L$ containing $K$. Suppose $\sigma : M \to M'$ is an isomorphism fixing $K$ pointwise and $p(x) \in M[x]$ is an irreducible polynomial. Let $\sigma(p)$ be the polynomial in $M'[x]$ obtained by applying $\sigma$ to all the coefficients of $p$. Suppose $\alpha$ and $\alpha'$ are elements of $L$ that are roots of the polynomials $p$ and $\sigma(p)$ respectively. Prove that $\alpha$ and $\alpha'$ have the same minimal polynomial over $K$.
4. Suppose $K$ is a field and $L$ is a finite normal extension of $K$. Let $G = \mathrm{Aut}(L/K)$, and let $M$ be the fixed field of $G$.
   (a) If $K$ is a perfect field (i.e., every irreducible polynomial over $K$ is separable), prove that $M = K$.
   (b) If $K$ has characteristic $p$, prove that for any $a \in M$, there exists some natural number $n$ such that $a^{p^n} \in K$. (Hint: Consider the minimal polynomial $f(x) \in K[x]$ of $a$, and prove that if $f$ is not linear, then $f(x) = g(x^p)$ for some polynomial $g$. Repeat.)
5. Prove that $\mathbb{Q}[x]/(\Phi_n(x))$ is a Galois extension of $\mathbb{Q}$, where $\Phi_n$ denotes the $n^{th}$ cyclotomic polynomial.
6. Suppose $K$ is a field and $f$ is a separable irreducible polynomial of degree $p$, where $p$ is prime. Let $L$ be the splitting field of $f$ over $K$. Prove that the Galois group of $L$ over $K$ contains a cyclic subgroup of order $p$.
7. Dummit and Foote, Section 14.2, # 4–5:
   Let $p$ be a prime, and let $f(x) = x^p - 2$.
   (a) Determine the elements of the Galois group of $f(x)$ over $\mathbb{Q}$.
   (b) Prove that the Galois group of $f(x)$ over $\mathbb{Q}$ is isomorphic to the group of matrices, $\{\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right] \mid a, b \in \mathbb{F}_p, a \neq 0\}$.
8. Dummit and Foote, Section 14.2, # 10 and 12:
   (a) Let $f(x) = x^8 - 3 \in \mathbb{Q}[x]$. Determine the Galois group of the splitting field of $f$ over $\mathbb{Q}$.
   (b) Let $g(x) = x^4 - 14x^2 + 9 \in \mathbb{Q}[x]$. Determine the Galois group of the splitting field of $g$ over $\mathbb{Q}$.
9. Dummit and Foote, Section 14.2, # 13:
   Prove that if the Galois group of the splitting field of a cubic over $\mathbb{Q}$ is the cyclic group of order 3, then all the roots of the cubic are real.