Math 259, Section 33: Honors Algebra III
Spring Quarter 2009
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Homework 7, Version 2
Due: WEDNESDAY, June 3, 2009

1. (*) Read Dummit and Foote, Sections 14.3–14.7.
2. (*) Dummit and Foote, Section 14.3, #1–7 and 10.
3. Dummit and Foote, Section 14.3, #8:

   Determine the splitting field of the polynomial $f(x) = x^p - x - a$ over $\mathbb{F}_p$, where $a \in \mathbb{F}_p$ and $a \neq 0$. Show explicitly that the Galois group is cyclic.
4. Dummit and Foote, Section 14.3, #11:

   Prove that $f(x) = x^{p^n} - x + 1$ is irreducible over $\mathbb{F}_p$ only when $n = 1$ or $n = p = 2$.
5. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Find an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.
6. Dummit and Foote, Section 14.4, #6:

   Read Proposition 24 and prove that $\mathbb{F}_p(x, y)/\mathbb{F}_p(x^p, y^p)$ is not a simple extension by explicitly exhibiting an infinite number of intermediate subfields.
7. Determine the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ and draw lattice diagrams of the intermediate subfields for $n = 13$ and $n = 16$.
8. Find the other roots of Cardano's two cubics:
   - $x^3 - 15x - 4$
   - $x^3 + 6x - 20$
9. (*) Dummit and Foote, Section 14.6. #1–14. (Please try some of these!)
10. Dummit and Foote, Section 14.6. #15:

    Let $p \in \mathbb{Z}$ be prime, and let $f(x) = x^4 + px + p \in \mathbb{Q}[x]$.

    (a) Show that $f(x)$ is irreducible for every prime p.

    (b) Show that if $p \neq 3, 5$, then the Galois group of $f(x)$ is S_4.

    (c) Show that if $p = 3$, then the Galois group of $f(x)$ is D_4.

    (d) Show that if $p = 5$, then the Galois group of $f(x)$ is C_4.