

Problem 1. True/False: write TRUE or FALSE on the line. Don't just write T and F, because if I can't tell them apart, it's wrong. (5 points each)

$$\int_{-1}^1 \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(1) - \sec^{-1}(-1) = 0 - \pi = -\pi$$

This is false for two reasons: the minor detail is that there should be an absolute value in the antiderivative, but the major detail is that the integrand doesn't exist at 0! You're integrating something infinite here. (Even if it did exist—which it could—the integrand is also odd, meaning if anything, it should be 0.)

_____ For every real x , $\tan(\tan^{-1}(x)) = x$.

This is true: the domain of \tan^{-1} is all reals, and it gives us an angle whose tangent is x . Then we take the tangent and get x back.

$$\int_0^{\pi/4} \frac{\sin^4(x)}{\cos^5(x)\tan^3(x)} = \sqrt{2} - 1.$$

True: canceling a lot of sines and cosines, we get $\int \sec x \tan x dx$, which is $\sec(\pi/4) - \sec(0)$.

$$\sin(\sec^{-1}(-13/5)) = 12/13.$$

True: $\sec^{-1} x$ gives us an angle in $[0, \pi]$ whose secant is $-13/5$. Drawing this triangle in the second quadrant gives the sine of that angle as $12/13$.

_____ Suppose that f and g are continuously differentiable functions, $f(0) = 0$, and $g(1) = 0$. Then

$$\int_0^1 f'(x)g(x)dx = - \int_0^1 f(x)g'(x)dx.$$

True: this is integration by parts with $g(x) = u, f'(x) = dv$. The uv term is $f(x)g(x)$ which gets evaluated at 0 and 1; it is 0 at both of these places.

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Problem 2. (10 points each) Calculate the following:

$$\int \sin(2x) \cos(4x) dx$$

Use the formula for $\sin(A) \cos(B)$ on the first page. This gives:

$$\begin{aligned} (1/2) \int (\sin(2x + 4x) + \sin(2x - 4x)) dx &= (1/2) \sin(6x) + \sin(-2x) \\ &= (1/2)(-\cos(6x)/6 - \cos(-2x)/(-2)) + C \end{aligned}$$

$$\int (\tan^{-4} x)(\sec^4 x) dx$$

We want to make a u -substitution for either tangent or secant. If we take out a $\sec^2(x) dx$, we get

$$\int (\tan^{-4} x)(\sec^2(x))(\sec^2 x) dx = \int (\tan^{-4} x)(1 + \tan^2(x))(\sec^2 x) dx$$

Now substitute $u = \tan x$ to get

$$\begin{aligned} \int (u^{-4}(1 + u^2) du) &= \int (u^{-4} + u^{-2}) du \\ &= u^{-3}/(-3) + u^{-1}/(-1) + C \\ &= (\tan(x))^{-3}/(-3) + (\tan(x))^{-1}/(-1) + C \end{aligned}$$

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Problem 3. (15 points) Calculate the following.

$$\int \frac{1+x^2}{\sqrt{1-x^2}} dx$$

Since we see a $1-x^2$, we should think to make a trig substitution with sine. Choose $x = \sin u$ and we get

$$\begin{aligned} \int \frac{1+x^2}{\sqrt{1-x^2}} dx &= \int \frac{1+\sin^2(u)}{\sqrt{1-\sin^2(u)}} \cos(u) du \\ &= \int (1+\sin^2(u)) du \end{aligned}$$

Now we use the formula for \sin^2 on the front page:

$$\begin{aligned} \int (1+\sin^2(u)) du &= \int (1+(1/2)(1-\cos(2u))) du \\ &= 3u/2 - (1/4)\sin(2u) + C \\ &= 3(\sin^{-1}(x))/2 - (1/4)\sin(2\sin^{-1}(x)) + C \end{aligned}$$

You could've left it like this, but an even better simplification (which no one did) would be to use $\sin(2y) = 2\sin(y)\cos(y)$, so

$$\sin(2\sin^{-1}(x)) = 2\sin(\sin^{-1}(x))\cos(\sin^{-1}(x)) = x\sqrt{1-x^2}.$$

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Problem 4. (15 points) a) We have calculated reduction formulas for integrals of powers of trigonometric functions, but such formulas exist for other functions too. Find a reduction formula for the following integral:

$$\int (\ln x)^n dx$$

Your answer should contain a smaller power of the natural logarithm.

Integrate by parts:

$$\begin{aligned} u &= (\ln x)^n & dv &= dx \\ du &= n(\ln x)^{n-1}(1/x) & v &= x \end{aligned}$$

which gives

$$\begin{aligned} \int (\ln x)^n dx &= x(\ln x)^n - n \int x(1/x)(\ln x)^{n-1} dx \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} dx \end{aligned}$$

This is a reduction formula, because it puts the integral into the same form, but with a smaller exponent. It was also possible to do this problem choosing $u = (\ln x)^{n-1}$, $dv = \ln x dx$, though it got messier this way. You get a different but still correct formula.

b)(5 points) Use your formula to calculate

$$\int (\ln x)^3 dx.$$

Iterating the formula 3 times, we have

$$\begin{aligned} \int (\ln x)^3 dx &= x(\ln x)^3 - 3 \int (\ln x)^2 dx \\ &= x(\ln x)^3 - 3(x(\ln x)^2 - 2 \int (\ln x)^1 dx) \\ &= x(\ln x)^3 - 3(x(\ln x)^2 - 2(x \ln x - \int 1 dx)) \\ &= x(\ln x)^3 - 3(x(\ln x)^2 - 2(x \ln x - x)) + C \end{aligned}$$

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Problem 5. a) (10 points) Find the partial fraction decomposition for the following rational function:

$$\frac{x^3 + 2x^2 - 1}{x^4 + x^3 + x^2}$$

The denominator factors into $x^2(x^2 + x + 1)$; the second quadratic does not factor. Note that x^2 does factor: $x^2 = x \cdot x$! Most people left it as a quadratic, which still works, so I didn't take off any points for it. We want to find A, B, C, D so that

$$\frac{x^3 + 2x^2 - 1}{x^4 + x^3 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + x + 1}$$

or equivalently

$$x^3 + 2x^2 - 1 = A(x^3 + x^2 + x) + B(x^2 + x + 1) + (Cx + D)(x^2)$$

Plugging in $x = 0$ or looking at the constant term gives $B = -1$. Looking at the x^3, x^2 , and x terms, we get

$$A + C = 1$$

$$A + B + D = 2$$

$$A + B = 0$$

The last equation now gives us $A = 1$, which we plug into the first two equations to get $C = 0$ and $D = 2$. Hence the decomposition is

$$\frac{x^3 + 2x^2 - 1}{x^4 + x^3 + x^2} = \frac{1}{x} + \frac{-1}{x^2} + \frac{2}{x^2 + x + 1}$$

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Problem 5 Continued.

b) (10 points) Use the decomposition from part (a) to calculate

$$\int \frac{x^3 + 2x^2 - 1}{x^4 + x^3 + x^2} dx$$

(You can put this on the previous page if you have room.)

From above, this simplifies to

$$\int \frac{x^3 + 2x^2 - 1}{x^4 + x^3 + x^2} dx = \int \left(\frac{1}{x} + \frac{-1}{x^2} + \frac{2}{x^2 + x + 1} \right) dx$$

The first two of these integrals are easy:

$$\int \frac{x^3 + 2x^2 - 1}{x^4 + x^3 + x^2} = \ln(|x|) + \frac{1}{x} + \int \frac{2}{x^2 + x + 1} dx$$

We can solve this last guy by completing the square:

$$\int \frac{2}{x^2 + x + (1/4) + (3/4)} dx = \int \frac{2}{(x + 1/2)^2 + (3/4)} dx$$

Let's first make the u -substitution $u = x + 1/2$, $du = dx$, to get

$$\int \frac{2}{(u)^2 + (3/4)} du$$

After a trig substitution, this is

$$(2/\sqrt{3/4}) \tan^{-1}(u/\sqrt{3/4}) + C = (2/\sqrt{3/4}) \tan^{-1}((x + 1/2)/\sqrt{3/4}) + C$$

Hence the final answer is:

$$\ln(|x|) + \frac{1}{x} + (2/\sqrt{3/4}) \tan^{-1}((x + 1/2)/\sqrt{3/4}) + C$$

Extra Credit! (1 pt., and your answer in chocolates) Choose the smallest positive integer that no one else will choose. _____