

Here are the solutions to the fifth homework. Notice that I *justify* everything that I claim is true. The idea in 10.4 is that you were supposed to use the examples we did in class to *explain* why the limits were what they were. If you just write an answer on the test with no explanation, then you will lose points.

10.4.6) $a_n = 3^n/4^n$ is of the form x^n , with $x = 3/4$. Since $|x| < 1$, by the example I from class, this sequence converges to 0. You could have also proved it directly that it goes to 0. If I choose $\varepsilon > 0$, I need to find an N such that, when $n > N$, $|(3/4)^n| < \varepsilon$. Solving this equation gives $n > \log(\varepsilon)/\log(3/4)$. Don't forget to switch the sign, since $\log(3/4) < 0$! Now if we choose $N > \log(\varepsilon)/\log(3/4)$, then $n > N$ implies $|(3/4)^n - 0| < \varepsilon$, which proves the limit is 0.

10.4.12) We can explicitly calculate this integral:

$$a_n = \int_{-n}^0 e^{2x} dx = (1/2)e^{2x} \Big|_{-n}^0 = 1/2 - e^{-2n}/2.$$

Because $e > 1$, $e^n \rightarrow \infty$ as $n \rightarrow \infty$, so $\lim_{n \rightarrow \infty} 1/(2e^n) = 0$. Thus $\lim_{n \rightarrow \infty} a_n = 0$.

10.4.24) If we right out the definition of $n!$, we find that

$$a_n = \frac{n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1}{2n} = \frac{(n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1}{2} = \frac{(n-1)!}{2}.$$

Since $(n-1)!$ goes to ∞ as $n \rightarrow \infty$, a_n blows up and this limit does not exist. We did not explicitly prove in class that factorials are not bounded, but this is pretty easy: to show that $b_n = n!$ is unbounded, choose any $M > 0$. Then $b_M = M! > M$, so M can't be an upper bound.

10.4.30) A number of people did this wrong by trying to take limits separately.

Following what we did for the example VII in 10.4, suppose $\lim_{n \rightarrow \infty} a_n = L$; then, by continuity, we can look at

$$\begin{aligned} \ln L &= \ln \lim_{n \rightarrow \infty} (1 + 1/n^2)^n = \lim_{n \rightarrow \infty} n \ln(1 + 1/n^2) \\ &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 1/n^2)}{1/n} \end{aligned}$$

We want to rewrite this to look like a derivative. Since $\ln 1 = 0$, we can rewrite this as

$$\lim_{n \rightarrow \infty} (1/n) \frac{\ln(1 + 1/n^2) - \ln 1}{1/n^2}$$

Let $h = 1/n^2$; then as $n \rightarrow \infty$, h goes to 0. So this is the same as the limit

$$\lim_{h \rightarrow 0} \sqrt{h} \frac{\ln(1 + h) - \ln 1}{h}$$

This is the product of two separate limits. If both of them exist, we can take each limit separately, but be careful! If one of them doesn't exist, and maybe we get $0 \cdot \infty$ or some other indeterminate form, we can't do this. Fortunately, in this case we can. We have

$$\lim_{h \rightarrow 0} \frac{\ln(1 + h) - \ln 1}{h} = 1$$

since this is the derivative of \ln at 1. Clearly

$$\lim_{h \rightarrow 0} \sqrt{h} = 0.$$

Since both limits exist, we can multiply them:

$$\lim_{h \rightarrow 0} \sqrt{h} \frac{\ln(1+h) - \ln 1}{h} = 0 \cdot 1 = 0.$$

It is also possible to do this with L'Hôpital's rule. Suppose $\lim_{n \rightarrow \infty} a_n = L$; then, by continuity, we can look at

$$\begin{aligned} \ln L &= \ln \lim_{n \rightarrow \infty} (1 + 1/n^2)^n = \lim_{n \rightarrow \infty} n \ln(1 + 1/n^2) \\ &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 1/n^2)}{1/n} \end{aligned}$$

If we replace n with x , both the top and bottom go to 0 as $x \rightarrow \infty$, so by L'Hôpital this is the same as

$$\lim_{x \rightarrow \infty} \frac{-1/x^3}{(-1/x^2)(1 + 1/x^2)} = \lim_{x \rightarrow \infty} \frac{1}{x(1 + 1/x^2)}.$$

This goes to 0 as $x \rightarrow \infty$, so $\ln L = 0$ and $L = 1$. Notice that, from what we know about L'Hôpital's rule now, the original limit is of the form 1^∞ , which is one of our indeterminate forms—this means you can't do it separately.

10.4.40) The trick is to rationalize the numerator:

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n) &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n} - n)\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} \end{aligned}$$

Factor an n out of the denominator and we get

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n} + 1}.$$

Since $1/n \rightarrow 0$, this simplifies to 2.

10.5.6) As $x \rightarrow a$, both the numerator and the denominator approach 0, so we get the indeterminate form $0/0$, meaning we can use L'Hôpital. If the limit exists, then

$$\lim_{x \rightarrow a} \frac{x - a}{x^n - a^n} = \lim_{x \rightarrow a} \frac{1}{nx^{n-1}} = \frac{1}{na^{n-1}}.$$

If $a = 0$ the limit doesn't exist, though this is a picky case.

10.5.10) As in the previous problem, the limits of the numerator and denominator separately are 0, so if the limit exists, it equals

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x^2 + x} = \lim_{x \rightarrow 0} \frac{e^x}{2x + 1} = 1.$$

A number of people tried to take a derivative again, and you get the wrong answer! Remember that you can only use L'Hôpital if you get $0/0$ or ∞/∞ .

10.5.24) The numerator approaches 0 as $x \rightarrow 0$, but the denominator approaches -1 . So this function is continuous at 0, and the limit is $0/-1 = 0$.

Extra problem 1: $a_n = 3 + 2/n$. We suspect the limit should be 3, since $2/n$ is getting small. Choose $\varepsilon > 0$; we want to see what we need to assume to have

$$|a_n - L| = |3 + 2/n - 3| < \varepsilon$$

This is the same as $2/n < \varepsilon$, which is the same as $n > 2/\varepsilon$. So, choose $N > 2/\varepsilon$; then if $n > N$, we have $n > 2/\varepsilon$, so $2/n < \varepsilon$ and $|a_n - L| < \varepsilon$, which proves that the limit exists.

Extra problem 2: $b_n = 7e^{-n}$. This is the same as $7/e^n$, which seems to get small as $n \rightarrow \infty$; so we should guess the limit is 0. Pick $\varepsilon > 0$, and try to figure out what happens when

$$|b_n - L| = |7e^{-n} - 0| < \varepsilon$$

This is the same as

$$7/\varepsilon < e^n,$$

which is the same as $\ln(7/\varepsilon) < n$. So choose $N > \ln(7/\varepsilon)$; then $7/\varepsilon < e^n$, which implies $|b_n - L| < \varepsilon$ as desired.