Spaces of non-degenerate maps between complex projective spaces

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Abstract

We study the space $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ of degree d algebraic maps $\mathbb{CP}^m \to \mathbb{CP}^n$, exhibiting homological stability as shown by Segal [26], Mostovoy [20], Farb–Wolfson [8], and others. In particular, we compute the $\mathbb{Q}$-cohomology ring explicitly in the case $m = 1$ and stably for when $m > 1$. In doing so, we develop a new approach to studying spaces of maps $X \to \mathbb{CP}^n$ by introducing subvarieties of non-degenerate functions which approximate the desired cohomologies both integrally and rationally in different ways. We also prove, when $m = n$, that the orbit space $\mathrm{Rat}_d(\mathbb{CP}^m, \mathbb{CP}^m)/\,\mathrm{PGL}_{m+1}(\mathbb{C})$ under the action on the target is $\mathbb{Q}$-acyclic up through dimension $d - 2$.

1 Introduction and main results

Any holomorphic map $f : \mathbb{CP}^m \to \mathbb{CP}^n$, $m \leq n$, can be represented as

$$f(z) = [f_0(z) : \cdots : f_n(z)] \tag{1.1}$$

where each $f_i \in \mathbb{C}[z_0, \ldots, z_m]$ is homogeneous of a common degree d and together have no common root. The degree of f is also characterized by a topological formula:

$$f^*(\omega_{\mathbb{CP}^n}) = d \cdot \omega_{\mathbb{CP}^m}, \tag{1.2}$$

where $\omega_X \in H^2(X; \mathbb{R})$ denotes the symplectic form of a Kähler manifold X. This representation (1.1) is unique up to scaling, so the space of all such maps $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ is a projective resultant complement of complex dimension $(n+1)\binom{m+d}{d} - 1$: see [5] for details.

In the $m = 1$ case, these functions are historically called *rational maps* and the notation

$$\mathrm{Rat}_d^n(\mathbb{C}) \coloneqq \mathrm{Hol}_d(\mathbb{CP}^1, \mathbb{CP}^n)$$

is used. In 1979, based on intuition from Morse theory, Segal [26] proved that the inclusion

$$\mathrm{Rat}_d^n(\mathbb{C}) \hookrightarrow \mathrm{Map}_d(\mathbb{CP}^1, \mathbb{CP}^n) \tag{1.3}$$

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is a homotopy equivalence through dimension $(2n-1)d$, where $\mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ is the space of continuous maps satisfying (1.2) equipped with the compact-open topology. This inspired many generalizations, for example extending the domain to genus $g \geq 1$ curves and the target to Grassmannians or toric varieties; see, e.g., [1, 3, 11, 12, 13, 15]. The work of Kozlowski–Yamaguchi [16] and Sasao [25] on linear maps, together with Segal-style stability due to Mostovoy [20] and Munguia-Villanueva [22], are apparently the only results when the domain has complex dimension greater than 1. In particular, Mostovoy [21] proved that inclusion of holomorphic maps into the space of continuous functions induces isomorphisms

$$H_i(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Z}) \to H_i(\mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Z})$$

for $2 \leq m \leq n$ and $i \leq d(2n-2m+1)-2$.

In 2015, Farb–Wolfson [8] showed that the Betti numbers for spaces of *based* rational maps $\mathbb{CP}^1 \to \mathbb{CP}^n$, written as $\mathrm{Rat}_d^n(\mathbb{C})^*$, are independent of the degree d. Their proof involved inducting on degree by "bringing in zeroes from infinity" via a (non-algebraic) map

$$\mathrm{Rat}_d^n(\mathbb{C})^* \times \mathbb{C}^{n+1} \to \mathrm{Rat}_{d+1}^n(\mathbb{C})^*$$

inducing isomorphisms on compactly supported rational cohomology. On the other hand, by observing that the embedding $\psi_d^{m,n} : \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) \to \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ given by

$$\psi_d^{m,n}(f)([z_0 : \cdots : z_m]) \coloneqq f([z_0^d : \cdots : z_m^d]) \tag{1.4}$$

induces the monomorphism $1 \mapsto d^m$ on fundamental groups when $m = n$, Yamaguchi [28] computed $\pi_1(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^m)) \cong \mathbb{Z}/(m+1)d^m\mathbb{Z}$. Our first result explains the invariance of degree observed by Farb–Wolfson using the homomorphisms induced by $\psi_d^n \coloneqq \psi_d^{1,n}$.

Theorem 1.1 (Sharp $\mathbb{Q}$-homological stability of $\mathrm{Rat}_d^n(\mathbb{C})$). *Fix $n, d \geq 1$. Then*

$$\psi_{d*}^n : H_i(\mathrm{Rat}_1^n(\mathbb{C}); \mathbb{Q}) \to H_i(\mathrm{Rat}_d^n(\mathbb{C}); \mathbb{Q})$$

is an isomorphism for all $i \geq 0$. In particular, $H_i(\mathrm{Rat}_d^n(\mathbb{C}); \mathbb{Q})$ does not depend on d.

Before proceeding, we recall that Kozlowski-Yamaguchi [16] showed the $\mathrm{U}(n+1)$-action induces a homotopy equivalence between $\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)$ and the complex projective Stiefel manifold $\mathrm{PW}_{m+1,n+1}(\mathbb{C})$ of orthonormal $(m+1)$-frames in $\mathbb{C}^{n+1}$. The latter is defined as

$$\mathrm{PW}_{m+1,n+1}(\mathbb{C}) \coloneqq \mathrm{U}(n+1)/(\Delta_{m+1} \times \mathrm{U}(n-m)) \tag{1.5}$$

where $\Delta_\ell \cong \mathrm{U}(1)$ is the center of $\mathrm{U}(\ell)$ and $\iota : \mathrm{PW}_{m+1,n+1}(\mathbb{C}) \hookrightarrow \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)$ given by

$$\iota([A])([z_0 : \cdots : z_m]) \coloneqq A \cdot [z_0 : \cdots : z_m : 0 \cdots : 0] \tag{1.6}$$

is a homotopy equivalence. The topology of Stiefel manifolds is well known: see, e.g., [7, 24].

The composition of (rational) equivalences $\psi_d^n \circ \iota$, alternatively thought of as inclusion of the $\mathrm{U}(n+1)$-orbit of the element $j_d^n \in \mathrm{Rat}_d^n(\mathbb{C})$ given by

$$j_d^n([x_0 : x_1]) = [x_0^d : x_1^d : 0 : \cdots : 0],$$

is the subject of our next result. We remark that the cohomology presentation which follows is a special case of Theorem 1.5 and Corollary 2.3 of Kallel–Salvatore [14].

Corollary 1.2 (Cohomology as a unitary orbit). *The map $\psi_d^n \circ \iota : \mathrm{PW}_{2,n+1}(\mathbb{C}) \to \mathrm{Rat}_d^n(\mathbb{C})$ is a rational homotopy equivalence. Accordingly, there is an isomorphism of graded $\mathbb{Q}$-algebras*

$$H^*(\mathrm{Rat}_d^n(\mathbb{C});\mathbb{Q}) \cong H^*(\mathrm{PW}_{2,n+1}(\mathbb{C});\mathbb{Q}) \cong \mathbb{Q}[y]/(y^n) \otimes \Lambda(x), \tag{1.7}$$

where $|y| = 2$ and $|x| = 2n+1$, for all $d \geq 1$.

In Section 2 we define subspaces of $\mathrm{Rat}_d^n(\mathbb{C})$ and, more generally, of $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ by maps whose image projectively span subspaces of a fixed dimension:

$$^r\mathrm{Rat}_d^n(\mathbb{C}) := \{f \in \mathrm{Rat}_d^n(\mathbb{C}) : \dim L_f = r\} \text{ and}$$
$$^r\mathrm{Hol}_d^{m,n}(\mathbb{C}) := \{f \in \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) : \dim L_f = r\},$$

where L_f is the intersection of all planes containing $f(\mathbb{CP}^m)$. The space $^m\mathrm{Hol}_d^{m,n}(\mathbb{C})$ arises naturally since $L_f = L_{\psi_d^{m,n}(f)}$ in an algebraically closed field, so the image of $\psi_d^{m,n}$ actually lands in this subvariety. We distinguish targets by writing

$$\begin{array}{ccc} ^m\mathrm{Hol}_d^{m,n}(\mathbb{C}) & \xhookrightarrow{\sigma_d^{m,n}} & \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \\ \phi_d^{m,n}\uparrow & \nearrow_{\psi_d^{m,n}} & \\ \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n). & & \end{array} \tag{1.8}$$

In the $m = 1$ case, where we shorten $\phi_d^n := \phi_d^{1,n}$ and $\sigma_d^n := \sigma_d^{1,n}$, we will show in Section 3 that all maps in the above diagram are rational homotopy equivalences.

Next, recall that a subset $Y \subset \mathbb{CP}^n$ is said to be *degenerate* if Y is contained in a hyperplane of $\mathbb{CP}^n$, and that a map $f : X \to \mathbb{CP}^n$ is said to be degenerate if the image $f(X)$ is degenerate. In this paper we will also study the subspaces of non-degenerate maps

$${}_\vartriangle\mathrm{Rat}_d^n(\mathbb{C}) := {}^n\mathrm{Rat}_d^n(\mathbb{C}) = \{f \in \mathrm{Rat}_d^n(\mathbb{C}) : f \text{ is non-degenerate}\} \text{ and}$$
$${}_\vartriangle\mathrm{Hol}_d^{m,n}(\mathbb{C}) := {}^n\mathrm{Hol}_d^{m,n}(\mathbb{C}) = \{f \in \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) : f \text{ is non-degenerate}\}.$$

where ${}_\vartriangle\mathrm{Rat}_d^n(\mathbb{C}) = \varnothing$ for all $n > d$ and more generally ${}_\vartriangle\mathrm{Hol}_d^{m,n}(\mathbb{C}) = \varnothing$ for all $n \geq \binom{m+d}{m}$.

Our main result concerns the subvarieties $^1\mathrm{Rat}_d^n(\mathbb{C})$ and ${}_\vartriangle\mathrm{Rat}_d^n(\mathbb{C})$, where the former should be thought of as the subspace of the "most degenerate" maps; whenever $d < n$, the variety $^d\mathrm{Rat}_d^n(\mathbb{C})$ of "least degenerate" maps is also of note. Part (b) of these results extend a result of Crawford (Theorem A of [4]) and are further extended in Theorem 1.5 of this paper.

Theorem 1.3 (Topology of non-degenerate rational maps). *Fix $n, d \geq 1$.*

(a) The inclusion $\sigma_d^n : {}^1\mathrm{Rat}_d^n(\mathbb{C}) \hookrightarrow \mathrm{Rat}_d^n(\mathbb{C})$ is a rational homotopy equivalence.

(b) If $d \geq n$ then the inclusion of non-degenerate maps induces an isomorphism

$$H_i({}_\vartriangle\mathrm{Rat}_d^n(\mathbb{C});\mathbb{Z}) \to H_i(\mathrm{Rat}_d^n(\mathbb{C});\mathbb{Z}),$$

for all $i \leq 2(d-n)$. If instead $d \leq n$, then inclusion induces an isomorphism

$$H_i({}^d\mathrm{Rat}_d^n(\mathbb{C});\mathbb{Z}) \to H_i(\mathrm{Rat}_d^n(\mathbb{C});\mathbb{Z}),$$

for all $i \leq 2(n-d)$. Moreover, in this case, there is an identification

$$^d\mathrm{Rat}_d^n(\mathbb{C}) \cong \mathrm{Hol}_1(\mathbb{CP}^d, \mathbb{CP}^n) \simeq \mathrm{PW}_{d+1,n+1}(\mathbb{C}). \tag{1.9}$$

Part (a) of Theorem 1.3 is clear: the subspace of "most degenerate" functions $\mathbb{CP}^1 \to \mathbb{CP}^n$ carry all the rational homological data of $\mathrm{Rat}_d^n(\mathbb{C})$. However, the latter results take some digesting. In words, the first part of Theorem 1.3(b) states that, as d grows with n fixed, the homology of the non-degenerate maps becomes a better approximation for the entire space of rational maps. On the other hand, as n grows with d fixed, the homology of $\mathrm{Rat}_d^n(\mathbb{C})$ becomes well-approximated by a projective Stiefel manifold, simply because most of the elements of the space can be identified in some precise sense with the standard rational normal curve.

Moreover, while ${}^1\mathrm{Rat}_d^n(\mathbb{C}) \hookrightarrow \mathrm{Rat}_d^n(\mathbb{C})$ is a rational homotopy equivalence, the inclusion fails to be an isomorphism integrally if $n > 1$; we will show in Section 2 that

$$\pi_1({}^1\mathrm{Rat}_d^n(\mathbb{C})) = \mathbb{Z}/d\mathbb{Z} \neq 0 = \pi_1(\mathrm{Rat}_d^n(\mathbb{C})).$$

Further, the bounds in Theorem 1.3(b) are sharp; in Section 2, for all $d > 2$, we compute

$$H_*({}_{\vartriangle}\mathrm{Rat}_d^2(\mathbb{C}); \mathbb{Q}) \cong H_*(\mathrm{Rat}_d^2(\mathbb{C}) \times S^{2d-3}; \mathbb{Q}),$$

a result which is originally due to Crawford [4].

As foreshadowed by our general notation, versions of these results hold when the domain $\mathbb{CP}^1$ is replaced by $\mathbb{CP}^m$ for $m > 1$. Difficulties arise because the modified Vassiliev [27] machinery of truncated resolutions as used by Mostovoy–Munguia-Villanueva [20, 21, 22] is the only method used thus far to understand spaces of polynomial maps $\mathbb{CP}^m \to Y$. In particular, results involving rational homology become isomorphisms in a stable range rather than outright equivalences:

Theorem 1.4 (**$\mathbb{Q}$-homological stability of $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$**). *Fix $1 < m \leq n$ and $1 \leq d$. Then*

$$\psi_{d\ *}^{m,n} : H_i(\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Q}) \to H_i(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Q}),$$

is an isomorphism for all $0 \leq i < d(2n - 2m + 1) - 1$. The stable cohomology is given by

$$H^*(\mathrm{PW}_{m+1,n+1}(\mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}[y]/(y^{n-m+1}) \otimes \Lambda(x_{2(n-m)+3}, \cdots, x_{2n+1}), \tag{1.10}$$

where $|y| = 2$ and $|x_{2j+1}| = 2j + 1$ for all $n - m < j \leq n$.

Accordingly, this theorem permits a similar set of results regarding the subspaces of most degenerate, non-degenerate, and least-degenerate holomorphic maps:

Theorem 1.5 (**Topology of non-degenerate holomorphic maps**). *Fix $1 \leq m \leq n$ and $1 \leq d$.*

(a) The homomorphism induced by inclusion of most-degenerate maps,

$$\sigma_{d\ *}^{m,n} : H_i({}^m\mathrm{Hol}_d^{m,n}(\mathbb{C}); \mathbb{Q}) \to H_i(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Q}),$$

is an isomorphism for all $i < d - 1$.

(b) If $\binom{m+d}{m} \geq n + 1$ then the inclusion of non-degenerate maps induces an isomorphism

$$H_i({}_{\vartriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}); \mathbb{Z}) \to H_i(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Z}),$$

for all $i \leq 2\binom{m+d}{m} - 2(n + 1)$. If instead $\binom{m+d}{m} \leq n + 1$, then there is an embedding

$$\mathrm{PW}_{\binom{m+d}{m},n+1}(\mathbb{C}) \to \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \tag{1.11}$$

inducing an integral homology isomorphism through dimension $2(n + 1) - 2\binom{m+d}{m}$.

In 1997, Milgram [17] studied the orbit space $\mathcal{X}_d := \mathrm{Rat}^1_d(\mathbb{C})/\,\mathrm{PGL}_2(\mathbb{C})$ under the action on the target $\mathbb{CP}^1$, exhibiting an isomorphism to the $\mathbb{Q}$-acyclic space of all projective classes of non-singular $d \times d$ Toeplitz matrices. In 2004, Yamaguchi [28] considered the more generalized space $\mathcal{X}^m_d := \mathrm{Rat}_d(\mathbb{CP}^m, \mathbb{CP}^m)/\,\mathrm{PGL}_{m+1}(\mathbb{C})$ and proved $\pi_1(\mathcal{X}^m_d) \cong \mathbb{Z}/d^m\mathbb{Z}$. Most recently, in 2019, Bergeron–Filom–Nariman [2] recovered Milgram's result and also proved a similar theorem for the more complicated quotient under the conjugation action.

In this paper we generalize Milgram's result, albeit in weaker form:

Corollary 1.6. *Fix $m > 1$ and $d \geq 3$. Then $H_i(\mathcal{X}^m_d; \mathbb{Q}) = 0$ for all $0 < i < d - 1$.*

In the context of maps $\mathbb{CP}^m \to \mathbb{CP}^n$, we ultimately have the following setup:

$$\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) \xrightarrow{\psi^{m,n}_d} \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \hookrightarrow \mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n),$$

where the composite map is a rational homotopy equivalence and the rightmost map is an integral homology equivalence in a stable range. In the case when $m = 1$, as suggested by the results of Farb–Wolfson [8], each map is a rational equivalence; if this were the case when $m > 1$, Equation 1.10 would describe $H^*(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Q})$ for any d, the induced maps in Theorem 1.5(a) would be outright isomorphisms, and $\mathcal{X}^m_d$ would be truly $\mathbb{Q}$-acyclic.

Although in principal the space $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ could have rational homology outside the stable range growing with d established by Mostovoy [20, 21], to date there is no calculation known to the author which has indicated the above maps are strictly stable rational isomorphisms. Indeed, there is some number-theoretic evidence discussed in Section 5 to suggest (or at least not disprove) that no such additional homology exists, using the machinery of étale cohomology theory introduced by Grothendieck to prove the Weil conjectures.

Our last result concerns counting the number of solutions to the equations defining the variety ${}_{\vartriangle}\mathrm{Rat}^n_d(\mathbb{C})$ over a fixed finite field $\mathbb{F}_q$, written $\#\,{}_{\vartriangle}\mathrm{Rat}^n_d(\mathbb{F}_q)$. We compute:

Theorem 1.7. *For any $1 < n \leq d$ and q a prime power,*

$$\#\,{}_{\vartriangle}\mathrm{Rat}^n_d(\mathbb{F}_q) = q^{2d+n(n-3)/2}\,(q^{d-1} - 1)\cdots(q^{d+1-n} - 1)(q^n - 1)(1 + q + \cdots + q^n) \qquad (1.12)$$

In particular, the probability of a random element $f \in \mathrm{Rat}^n_d(\mathbb{F}_q)$ being non-degenerate is

$$(1 - q^{-(d-1)})\cdots(1 - q^{-(d+1-n)}). \qquad (1.13)$$

In Section 2 we establish notation used throughout the paper and define the subvarieties ${}_{\vartriangle}\mathrm{Hol}^{m,n}_d(\mathbb{C})$ and ${}^r\mathrm{Hol}^{m,n}_d(\mathbb{C})$, proving part (b) of Theorems 1.3 and 1.5 on the way. Section 3 is devoted to Theorems 1.1 and 1.3 (a) on rational maps, while Section 4 studies the analogous Theorems 1.4, 1.5 (a), and 1.6 for holomorphic maps $\mathbb{CP}^m \to \mathbb{CP}^n$. Section 5 discusses the interplay of arithmetic and topology, including Theorem 1.7 and homological density.

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2 Preliminaries, notation, and setup

In this section we proceed in full generality, considering holomorphic maps $\mathbb{CP}^m \to \mathbb{CP}^n$ for any $1 \le m \le n$, since there is nothing gained by specifying the $m = 1$ case here. Throughout the paper we make use of the Poincaré polynomial $P_t(X)$ associated to a space X, defined as

$$P_t(X) \coloneqq \sum_{i=0}^{\infty} \dim_{\mathbb{Q}} H^i(X;\mathbb{Q})\, t^i.$$

We will also write $X \simeq_{\mathbb{Q}} Y$ to denote a rational homotopy equivalence, meaning a map of simply-connected spaces inducing isomorphisms on homotopy groups after tensoring with $\mathbb{Q}$.

Recall that the equivalence $\mathrm{PW}_{m+1,n+1}(\mathbb{C}) \hookrightarrow \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)$ induced by the orbit of the natural $\mathrm{U}(n+1)$-action can be thought of as an equivalence of bundles via Gram-Schmidt:

$$\begin{array}{ccccc} & & \mathrm{PGL}_{m+1}(\mathbb{C}) & \hookrightarrow & \mathrm{Hol}_1(\mathbb{CP}^m,\mathbb{CP}^n) \\ & \overset{\simeq}{\nearrow} & & \nearrow & \downarrow \\ \mathrm{PU}(m+1) & \hookrightarrow & \mathrm{PW}_{m+1,n+1}(\mathbb{C}) & & \\ & & & \searrow & \\ & & & & \mathrm{Gr}(\mathbb{CP}^m,\mathbb{CP}^n), \end{array} \tag{2.1}$$

where the projections assign an orthonormal frame $[A]$ to its span and a map f to its image. Our objective is to generalize this setup to nonlinear maps. While the author originally believed what follows to be a novel construction, similar work was carried out in the 1993 thesis of Crawford [4] using homotopy theory and with a focus on the $n = 2$ case.

For fixed $1 \le m \le n$ and $d > 1$ we define an increasing family of locally closed subspaces:

$${}^{r}_{\blacktriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) \coloneqq \{f \in \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \,:\, \dim L_f \le r\}, \tag{2.2}$$

where L_f is the projective span of the image $f(\mathbb{CP}^m)$. Equivalently, we can define ${}^{r}_{\blacktriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C})$ as the image of the following map:

$$\begin{array}{c} Z \coloneqq \{(f,P) \in \mathrm{Hol}_d(\mathbb{CP}^m,\mathbb{CP}^n) \times \mathrm{Gr}(\mathbb{CP}^r,\mathbb{CP}^n) : p \circ f = 0 \text{ for all } p \in \mathrm{I}_{\mathbb{CP}^n}(P)\} \\ \big\downarrow \\ \mathrm{Hol}_d(\mathbb{CP}^m,\mathbb{CP}^n), \end{array}$$

where $\mathrm{I}_{\mathbb{CP}^n}(P)$ is the homogeneous ideal defining the plane P, generated by $n-r$ linear polynomials, and the vertical map forgets P. It is often convenient for calculations to think of the Grassmannian as the space of rank $r+1$ Hermitian projections M in $\mathbb{C}^{n+1}$, where a plane $P \in \mathrm{Gr}(\mathbb{CP}^r, \mathbb{CP}^n)$ corresponds (uniquely) to the column space of M:

$$\mathrm{Gr}(\mathbb{CP}^r,\mathbb{CP}^n) \cong \{M \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{C}) : M = M^2 = M^\dagger \text{ and } \mathrm{Trace}(M) = r+1\}.$$

With this setup, the condition on $(f, M) \in Z$ is $\operatorname{im} f \subset \operatorname{im} M = \ker(\mathrm{id} - M)$. We write

$${}^{r}\mathrm{Hol}_d^{m,n}(\mathbb{C}) \coloneqq {}^{r}_{\blacktriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) - {}^{r-1}_{\blacktriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) = \{f \in \mathrm{Hol}_d(\mathbb{CP}^m,\mathbb{CP}^n) : \dim L_f = r\} \tag{2.3}$$

to obtain a stratification of $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ by varying r. Hence there is a spectral sequence

$$\left\{E_k^{r,s},\ d_k : E_k^{r,s} \to E_k^{r+k,s+1-k}\right\} \implies H_c^{r+s}(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Z}) \tag{2.4}$$

with the E_1 term given by the cohomology of the strata $E_1^{r,s} := H_c^{r+s}({}^r\mathrm{Hol}_d^{m,n}(\mathbb{C}); \mathbb{Z})$. In order to study these strata, we introduce notation for the particular case when $n = r$:

$$_{\vartriangle}\mathrm{Hol}_d^{m,r}(\mathbb{C}) := {}^r\mathrm{Hol}_d^{m,r}(\mathbb{C}) = \{f \in \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^r) : f \text{ is non-degenerate}\}. \tag{2.5}$$

This space can be identified with the fiber of the map

$$\rho_d : {}^r\mathrm{Hol}_d^{m,n}(\mathbb{C}) \to \mathrm{Gr}(\mathbb{CP}^r, \mathbb{CP}^n),$$

given by $\rho_d(f) = L_f$, over a fixed $\mathbb{CP}^r \subset \mathbb{CP}^n$. Indeed, ρ_d is a locally trivial fiber bundle associated to the principle $\mathrm{Stab}(\mathbb{CP}^r)$-bundle which defines the Grassmannian as a homogeneous space. All told, we have a spectral sequence computing the compactly supported cohomology of $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ in terms of the spaces of non-degenerate maps:

$$E_1^{r,s} := H_c^{r+s}({}_{\vartriangle}\mathrm{Hol}_d^{m,r}(\mathbb{C}) \times_{\mathrm{Stab}(\mathbb{CP}^r)} \mathrm{PGL}_{n+1}(\mathbb{C}); \mathbb{Z}) \implies H_c^{r+s}(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Z}) \tag{2.6}$$

Next we note that ${}_{\vartriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) = \varnothing$ if $n < m$ and that ${}_{\vartriangle}\mathrm{Hol}_d^{m,m}(\mathbb{C}) = \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^m)$. More generally recall that, for an algebraic variety X, a rational map $X \to \mathbb{CP}^n$ can be thought of as a choice of $n+1$ sections $\sigma_0, \ldots, \sigma_n \in H^0(X; L)$ of a line bundle L defined by

$$f(x) = [\sigma_0(x) : \cdots : \sigma_n(x)],$$

where the degree d of L corresponds to the degree of f. In fact there is a bijection

$$\begin{gathered}\{V \leq H^0(X; L) : \dim V = n+1 \text{ and } V \text{ has no common zeroes}\} \\ \longleftrightarrow \{\text{non-degenerate degree } d \text{ morphisms } X \to \mathbb{CP}^n\} / \mathrm{Aut}(\mathbb{CP}^n).\end{gathered}$$

Because the degree d monomials in $m+1$ variables can be identified as a basis for the space of global sections $H^0(\mathbb{CP}^m; \mathcal{O}_{\mathbb{CP}^m}(d))$, this bijection gives rise to a Zariski-open embedding

$${}_{\vartriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) / \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Gr}(n+1, \tbinom{m+d}{d}). \tag{2.7}$$

Hence ${}_{\vartriangle}\mathrm{Hol}_d^{m,r}(\mathbb{C}) = \varnothing$ if $r \geq \binom{m+d}{d}$. We also can count dimensions of the associated spaces:

$$\begin{gathered}\dim_{\mathbb{C}} {}_{\vartriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) = (n+1)\tbinom{m+d}{d} - 1 \text{ and} \\ \dim_{\mathbb{C}} {}^r\mathrm{Hol}_d^{m,n}(\mathbb{C}) = (r+1)\left((n-r) + \tbinom{m+d}{d}\right) - 1.\end{gathered} \tag{2.8}$$

Next, recall that the *rational normal curve* is the smooth curve $\nu_d : \mathbb{CP}^1 \to \mathbb{CP}^d$ given by

$$\nu_d([x_0 : x_1]) = [x_0^d : x_0^{d-1}x_1 : \cdots : x_0 x_1^{d-1} : x_1^d]. \tag{2.9}$$

Moreover, every irreducible degree d non-degenerate curve in $\mathbb{CP}^d$ is $\mathrm{PGL}_{d+1}(\mathbb{C})$-conjugate to the standard rational normal curve. By this uniqueness property, if $n \geq d$ we see that

$$\begin{gathered}{}_{\vartriangle}\mathrm{Rat}_d^d(\mathbb{C}) \cong \mathrm{PGL}_{d+1}(\mathbb{C}) \text{ and} \\ {}^d\mathrm{Rat}_d^n(\mathbb{C}) \cong \mathrm{Hol}_1(\mathbb{CP}^d, \mathbb{CP}^n) \simeq \mathrm{PW}_{d+1,n+1}(\mathbb{C}).\end{gathered} \tag{2.10}$$

This statement generalizes via the Veronese map $\nu_d^m : \mathbb{P}(\mathbb{C}^{m+1}) \to \mathbb{P}(\operatorname{Sym}^d \mathbb{C}^{m+1})$, given by sending $[x_0 : \cdots : x_m]$ to all possible monomials of total degree d, which is also unique up to automorphism. Hence we have the identification

$$ {}^{\binom{m+d}{d}-1}\mathrm{Hol}_d^{m,n}(\mathbb{C}) \cong \mathrm{Hol}_1(\mathbb{CP}^{\binom{m+d}{d}-1}, \mathbb{CP}^n) \tag{2.11}$$

whenever $\binom{m+d}{d} \leq n+1$.

Remark 2.1. Away from edge cases the ${}_{\vartriangle}\mathrm{Hol}_d^{m,r}(\mathbb{C})$ are more difficult to understand without a more sophisticated analysis, although minimal-degree non-degenerate varieties have been classified due to Bertini: see, for example, [10].

Proof of Theorems 1.3 and 1.5 (b). The desired ranges come from the dimension counts (2.8) and the identifications (2.10, 2.11), together with Poincaré duality and the spectral sequence (2.6) computing the compactly supported homology of $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$. □

Corollary 2.2. *For any $m, d \geq 1$ and $n < \binom{m+d}{m}$, we have ${}_{\vartriangle}\mathrm{Hol}_d^{m,n}(\mathbb{C}) \neq \varnothing$ for any $n < \binom{m+d}{m}$.*

With these preliminaries established, the maps $\psi_d^{m,n}$ and $\phi_d^{m,n}$ from (1.8) given by

$$\phi_d^{m,n} : \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) \to {}^m\mathrm{Hol}_d^{m,n}(\mathbb{C}) \text{ and}$$
$$\psi_d^{m,n} : \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) \to \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \text{ via}$$
$$\psi_d^{m,n}(f)([x_0 : \cdots : x_m]) \coloneqq \phi_d^{m,n}(f)([x_0 : \cdots : x_m]) \coloneqq f([x_0^d : \cdots : x_m^d])$$

together constitute a morphism of bundles:

$$\begin{array}{ccccc}
 & & \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^m) & \hookrightarrow & {}^m\mathrm{Hol}_d^{m,n}(\mathbb{C}) \\
 & \nearrow^{\psi_d^{m,m}} & & \nearrow_{\phi_d^{m,n}} & \downarrow{\scriptstyle \rho_d} \\
\mathrm{PGL}_{m+1}(\mathbb{C}) & \hookrightarrow & \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) & & \\
 & & & \searrow_{\rho_1} & \\
 & & & & \mathrm{Gr}(\mathbb{CP}^m, \mathbb{CP}^n).
\end{array} \tag{2.12}$$

Therefore much of this paper will rely on the rational homological data carried by $\psi_d^{m,m}$.

We conclude with a computation of fundamental groups to confirm that the inclusion ${}^m\mathrm{Hol}_d^{m,n}(\mathbb{C}) \hookrightarrow \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ fails to be a homotopy equivalence when $m < n$ and $d > 1$.

Proposition 2.3. *Fix $1 \leq m < n$ and $1 \leq d$. Then $\pi_1({}^m\mathrm{Hol}_d^{m,n}(\mathbb{C})) \cong \mathbb{Z}/d^m\mathbb{Z}$.*

Proof. Since $\pi_1(\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)) = 0$ whenever $m < n$ by [28], the result follows directly by naturality of the long exact sequence in homotopy groups applied to $\phi_d^{m,n}$:

$$\begin{array}{ccccccc}
\overbrace{\pi_2(\mathrm{Gr}(\mathbb{CP}^m, \mathbb{CP}^n))}^{=\mathbb{Z}} & \longrightarrow & \overbrace{\pi_1(\mathrm{Hol}_d^{m,m}(\mathbb{C}))}^{=\mathbb{Z}/(m+1)d^m\mathbb{Z}} & \longrightarrow & \pi_1({}^m\mathrm{Hol}_d^{m,n}(\mathbb{C})) & \longrightarrow & \overbrace{\pi_1(\mathrm{Gr}(\mathbb{CP}^m, \mathbb{CP}^n))}^{=0} \\
\uparrow & & \uparrow{\scriptstyle 1 \mapsto d^m} & & \uparrow & & \| \\
\underbrace{\pi_2(\mathrm{Gr}(\mathbb{CP}^m, \mathbb{CP}^n))}_{=\mathbb{Z}} & \longrightarrow & \underbrace{\pi_1(\mathrm{PGL}_{m+1}(\mathbb{C}))}_{=\mathbb{Z}/(m+1)\mathbb{Z}} & \longrightarrow & \underbrace{\pi_1(\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n))}_{=0} & \longrightarrow & \underbrace{\pi_1(\mathrm{Gr}(\mathbb{CP}^m, \mathbb{CP}^n))}_{=0}.
\end{array}$$

□

3 Rational maps from $\mathbb{CP}^1$

The objective of this section is to prove Theorems 1.1 and 1.3(a). We will do so by showing that both ψ_d^n and ϕ_d^n in the commutative diagram

$$\begin{array}{ccc} {}^1\mathrm{Rat}_d^n(\mathbb{C}) & \xhookrightarrow{\sigma_d^n} & \mathrm{Rat}_d^n(\mathbb{C}) \\ \phi_d^n \uparrow & \nearrow \psi_d^n & \\ \mathrm{Rat}_1^n(\mathbb{C}). & & \end{array} \tag{3.1}$$

are rational homotopy equivalences. Note that some of the results and methods in this section can be found in the literature, for example [14, 16], however we include the proofs both for continuity and for their instructive use toward later calculations.

Proposition 3.1 (Kozlowski–Yamaguchi [16])**.** *For any $n \geq 1$,*

$$P_t(\mathrm{Rat}_1^n(\mathbb{C})) = (1 + t^2 + \cdots + t^{2(n-1)})(1 + t^{2n+1}) \tag{3.2}$$

Proof. We begin with the evaluation fiber bundle used by Segal [26] and many others

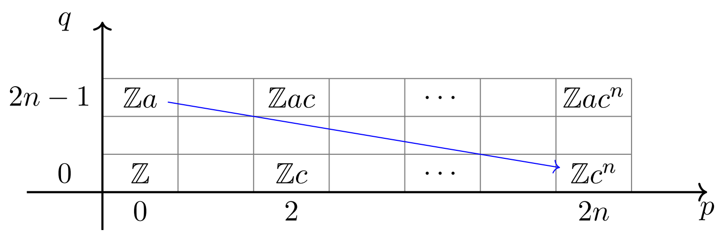
$$S^{2n-1} \simeq \mathrm{Rat}_1^n(\mathbb{C})^* \to \mathrm{Rat}_1^n(\mathbb{C}) \to \mathbb{CP}^n,$$

whose Serre spectral sequence is determined by the E_{2n} transgression. We can deformation retract the total space fiberwise to a space E wherein the fiber F_ℓ over a line $\ell \in \mathbb{CP}^n$ is $\mathbb{C}^{n+1}/\ell - 0$. In fact, E is isomorphic to $T\mathbb{CP}^n - 0$, where 0 is the zero section, via the map

$$\tau : E \hookrightarrow \mathrm{Hom}(\gamma, \gamma^\perp) \cong T\mathbb{CP}^n$$

$$\tau(v)(\ell) = v,$$

where γ is the tautological bundle. Thus we can compute the transgression in terms of the top Chern class $c_n(T\mathbb{CP}^n) = (n+1)c^n$, where c is a suitable generator for $H^2(\mathbb{CP}^n; \mathbb{Z})$:



In particular, $H_{2n-1}(\mathrm{Rat}_1^n(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}/(n+1)\mathbb{Z}$ is the only torsion term. □

Remark 3.2. Attempting to write down a section of $\mathrm{Rat}_1^n(\mathbb{C}) \to \mathbb{CP}^n$ amounts to making a continuous choice of a line distinct to the one determined by $f(\infty)$. Identifying these linear maps with matrices, the projection onto the base takes the simple form

$$\begin{pmatrix} a_0 & b_0 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \mapsto [a_0 : \cdots : a_n].$$

No such section exists since the differential is nonzero. We will show in Section 4 that the bundle one uses to generalize this proof to $\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)$ sends a full-rank $(n+1)\times(m+1)$ matrix to its first m columns. As we have just seen in the previous proof, the work boils down to computing a single transgression whose non-triviality is tantamount to the impossibility of continuously picking a line distinct from a varying m-plane in $\mathbb{C}^{n+1}$.

Proof of Theorem 1.1. In the context of continuous based functions $\mathbb{CP}^1 \to \mathbb{CP}^n$, we define

$$\theta_d^{n\prime} : \mathrm{Map}_1^*(\mathbb{CP}^1, \mathbb{CP}^n) \to \mathrm{Map}_d^*(\mathbb{CP}^1, \mathbb{CP}^n)$$
$$\theta_d^{n\prime}(f)([x_0 : x_1]) = f([x_0^d : x_1^d]).$$

Møller [19] showed that $\mathrm{Map}_d^*(\mathbb{CP}^1, \mathbb{CP}^n)$ is $2(n-1)$-connected, that the homotopy group

$$\pi_{2n-1}(\mathrm{Map}_d^*(\mathbb{CP}^1, \mathbb{CP}^n)) \cong \mathbb{Z},$$

and that ${\theta_d^{n\prime}}_*$ is multiplication by d on π_{2n-1}. On the other hand, ψ_d^n induces a bundle map:

$$\begin{array}{ccccc} & & \mathrm{Rat}_d^n(\mathbb{C})^* & \hookrightarrow & \mathrm{Rat}_d^n(\mathbb{C}) \\ & \nearrow^{\psi_d^{n\prime}} & & \nearrow^{\psi_d^n} & \downarrow {\scriptstyle f\mapsto f(\infty)} \\ S^{2n-1} \simeq \mathrm{Rat}_1^n(\mathbb{C})^* & \hookrightarrow & \mathrm{Rat}_1^n(\mathbb{C}) & & \\ & & & \searrow_{f\mapsto f(\infty)} & \\ & & & & \mathbb{CP}^n, \end{array} \tag{3.3}$$

where $\psi_d^{n\prime}$ is the restriction to the based rational maps, as studied in Proposition 2.2 of [14]. We apply Segal's stability result [26] for $\mathrm{Rat}_d^n(\mathbb{C})^* \hookrightarrow \mathrm{Map}_d^*(\mathbb{CP}^1, \mathbb{CP}^n)$ to see that

$${\psi_d^{n\prime}}_* : H_{2n-1}(\mathrm{Rat}_1^n(\mathbb{C})^*; \mathbb{Z}) \to H_{2n-1}(\mathrm{Rat}_d^n(\mathbb{C})^*; \mathbb{Z}) \tag{3.4}$$

is multiplication by d and therefore an isomorphism after tensoring with $\mathbb{Q}$:

$$\begin{array}{ccc} H_{2n-1}(\mathrm{Rat}_d^n(\mathbb{C})^*; \mathbb{Z}) & \xrightarrow{\cong \text{ by Segal}} & H_{2n-1}(\mathrm{Map}_d^*(\mathbb{CP}^1, \mathbb{CP}^n); \mathbb{Z}) \cong \mathbb{Z} \\ {\scriptstyle {\psi_d^{n\prime}}_*}\uparrow & & {\scriptstyle {\theta_d^{n\prime}}_*}\uparrow{\scriptstyle 1\mapsto d} \\ H_{2n-1}(\mathrm{Rat}_1^n(\mathbb{C})^*; \mathbb{Z}) & \xrightarrow[\cong \text{ by Segal}]{} & H_{2n-1}(\mathrm{Map}_1^*(\mathbb{CP}^1, \mathbb{CP}^n); \mathbb{Z}) \cong \mathbb{Z}. \end{array}$$

Since $H^*(\mathrm{Rat}_d^n(\mathbb{C})^*; \mathbb{Q}) \cong H^*(S^{2n-1}; \mathbb{Q})$ for all d, as shown in [3] and [8], the bundles in (3.3) induce spectral sequences whose only differential is the E^{2n} transgression. The connecting homomorphism (3.4) between the spectral sequences is an isomorphism, so they must coincide with rational coefficients. In fact we have shown that the $E^{2n}(\mathrm{Rat}_d^n(\mathbb{C}))$ transgression can be identified with multiplication by $(n+1)d$:

$$\begin{array}{ccc} H_{2n}(\mathbb{CP}^n; \mathbb{Z}) & \xrightarrow{1\mapsto(n+1)d} & H_{2n-1}(\mathrm{Rat}_d^n(\mathbb{C})^*; \mathbb{Z}) \\ \| & & \uparrow{\scriptstyle 1\mapsto d} \\ H_{2n}(\mathbb{CP}^n; \mathbb{Z}) & \xrightarrow[1\mapsto n+1]{} & H_{2n-1}(\mathrm{Rat}_1^n(\mathbb{C})^*; \mathbb{Z}). \end{array}$$

For more details, see [14] which computes $H_*(\mathrm{Rat}_d^n(\mathbb{C}))$ with arbitrary field coefficients. □

Remark 3.3. The previous proof can be done entirely in the holomorphic category, much in the style of Lemmas 3.1 and 3.10 in [2]; note that the map ψ_d in question traces out the $\mathrm{PGL}_{n+1}(\mathbb{C})$-orbit via post-composition of the element $j_d^n \in \mathrm{Rat}_d^n(\mathbb{C})$ given by

$$j_d^n([x_0 : x_1]) \coloneqq [x_0^d : x_1^d : 0 : \cdots : 0].$$

We have chosen to include the previous proof due to its thematic suitability for this paper.

Lemma 3.4. *Fix integers $d, n \geq 1$. Then the induced map*

$$\phi_{d*} : H_i(\mathrm{Rat}_1^n(\mathbb{C}); \mathbb{Q}) \to H_i({}^1\mathrm{Rat}_d^n(\mathbb{C}); \mathbb{Q})$$

is an isomorphism for all $i \geq 0$.

Proof. As in the proof of Theorem 1.1, we will show the bundle morphism

$$\begin{array}{ccccc} & & \mathrm{Rat}_d^1(\mathbb{C}) & \hookrightarrow & {}^1\mathrm{Rat}_d^n(\mathbb{C}) \\ & \overset{\psi_d^n}{\nearrow} & & \overset{\phi_d^n}{\nearrow} & \Big\downarrow \rho_d \\ \mathrm{PGL}_2(\mathbb{C}) & \hookrightarrow & \mathrm{Rat}_1^n(\mathbb{C}) & & \\ & & & \underset{\rho_1}{\searrow} & \\ & & & & \mathrm{Gr}(\mathbb{CP}^1, \mathbb{CP}^n). \end{array} \tag{3.5}$$

induces a natural isomorphism between the Serre spectral sequences after tensoring with $\mathbb{Q}$. To be explicit, both sequences are determined by their E^4 transgression. The connecting map

$$\psi_{d*}^n : E_{0,3}^4(\mathrm{Rat}_1^n(\mathbb{C})) = H_3(\mathrm{PGL}_2(\mathbb{C}); \mathbb{Q}) \to H_3(\mathrm{Rat}_d^1(\mathbb{C}); \mathbb{Q}) = E_{0,3}^4({}^1\mathrm{Rat}_d^n(\mathbb{C}))$$

between the spectral sequences is an isomorphism by Theorem 1.1, so the differentials in each sequence have the same rank. The result follows by naturality. □

Proof of Theorem 1.3. We have shown that two of the three maps, ψ_d^n and ϕ_d^n, in the commutative diagram (3.1) are rational equivalences. Hence σ_d^n is also a rational equivalence. □

One can now compute the groups $H^i({}_{\vartriangle}\mathrm{Rat}_d^n(\mathbb{C}); \mathbb{Q})$ inductively via the spectral sequence (2.6), where the first column and the abutted cohomology have Betti numbers given by (3.2). In particular, it is easy to show the following corollary simply using the long exact sequence in compactly supported cohomology:

Corollary 3.5. *For any $d > 2$,*

$$H_*({}_{\vartriangle}\mathrm{Rat}_d^2(\mathbb{C}); \mathbb{Q}) \cong H_*(\mathrm{Rat}_d^2(\mathbb{C}) \times S^{2d-3}; \mathbb{Q}) \cong H_*(\mathrm{PW}_{2,3}(\mathbb{C}) \times S^{2d-3}; \mathbb{Q}). \tag{3.6}$$

We note that this result was first proved, via different methods, in 1993 by Crawford [4]. In Section 5 we discuss how arithmetic predicts the Betti numbers of ${}_{\vartriangle}\mathrm{Rat}_d^n(\mathbb{C})$ for $n > 2$.

4 Holomorphic maps from $\mathbb{CP}^m$, $m > 1$

Unfortunately, the convenient equivalence $S^{2n-1} \simeq_{\mathbb{Q}} \mathrm{Rat}_d^n(\mathbb{C})^*$ is lost when the domain has complex dimension $m > 1$. In fact, from the results proved in this Section, one can show that

$$H_i(\mathrm{Map}_d^*(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Q}) \cong H_i(S^{2n-1} \times \cdots \times S^{2(n-m)+1}; \mathbb{Q}); \tag{4.1}$$

however, as this isomorphism is not the focus of this paper, we will omit details. In order to proceed, we must generalize the $m = 1$ technique of the evaluation fibration in a way which respects the maps $\psi_d^{m,n}$ and $\phi_d^{m,n}$. To that end, fix once and for all $1 < m \leq n$.

An important observation is that evaluating $f : \mathbb{CP}^1 \to \mathbb{CP}^n$ at $\infty \in \mathbb{CP}^1$ is the same as restricting f to a particular $\mathbb{CP}^0 \subset \mathbb{CP}^1$. Therefore the generalized idea for maps $\mathbb{CP}^m \to \mathbb{CP}^n$ becomes restricting to a fixed hyperplane $\mathbb{CP}^{m-1} \subset \mathbb{CP}^m$. The advantage of what follows will be that the fiber will have a discriminant which is easier to describe than $\mathrm{Hol}_d^*(\mathbb{CP}^m, \mathbb{CP}^n)$, as exploited by Mostovoy [20].

To study such a restriction map we should also introduce notation to describe its fibers. Given a holomorphic map $f : \mathbb{CP}^{m-1} \to \mathbb{CP}^n$ and the usual inclusion $i : \mathbb{CP}^{m-1} \hookrightarrow \mathbb{CP}^m$ into the first m coordinates, setting $d = \deg(f)$, we define a subvariety of maps as follows:

$$\mathrm{Hol}_f(\mathbb{C}) \coloneqq \{g \in \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \,:\, g \circ i = f\}.$$

The space $\mathrm{Hol}_f(\mathbb{C})$ is clearly nonempty and can be identified as a quasi-affine variety of dimension $\binom{m+d-1}{d-1}(n+1)$. We use the same notation in the context of continuous maps:

$$\mathrm{Map}_f \coloneqq \{g \in \mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n) \,:\, g \circ i = f\}.$$

These spaces come equipped with a natural basepoint, namely f.

As explained more carefully in [19], the inclusion i induces a fibration

$$\mathrm{Map}_f \hookrightarrow \mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n) \twoheadrightarrow \mathrm{Map}_d(\mathbb{CP}^{m-1}, \mathbb{CP}^n).$$

Moreover, since $\mathbb{CP}^m = \mathbb{CP}^{m-1} \cup_h B^{2m}$ can be identified with the mapping cone of the Hopf map $h : S^{2m-1} \to \mathbb{CP}^{m-1}$, it follows that there is an equivalence $\mathrm{Map}_f \simeq \Omega^{2m}\mathbb{CP}^n$. In the holomorphic category, the restriction map

$$R : \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \twoheadrightarrow \mathrm{Hol}_d(\mathbb{CP}^{m-1}, \mathbb{CP}^n)$$

is not necessarily a fiber bundle. However, Thom's isotopy lemma can be applied to produce a finite filtration $F_0 \subset F_1 \subset \cdots \subset F_q = \mathrm{Hol}_d(\mathbb{CP}^{m-1}, \mathbb{CP}^n)$ wherein R is a fiber bundle over each connected component of the strata $F_{i+1} - F_i$. : see [20] for details. As such, we will conduct our analysis in the continuous category before returning to holomorphic maps.

First we mention a known fact, which is difficult to find in the literature, for completeness.

Lemma 4.1. *Fix $1 < m \leq n$. Then $\pi_{2(n-m)+1}(\Omega^{2m}\mathbb{CP}^n) \cong \mathbb{Z}$ and $\Omega^{2m}\mathbb{CP}^n \simeq_{\mathbb{Q}} S^{2(n-m)+1}$.*

Sketch of proof. The equivalence comes from the Hopf map extended to a sequence

$$\cdots \to \Omega^{2m}S^{2n+1} \to \Omega^{2m}\mathbb{CP}^n \to \Omega^{2m-1}S^1 \to \cdots \to \Omega\mathbb{CP}^n \to S^1 \to S^{2n+1} \to \mathbb{CP}^n,$$

where every two consecutive maps is a fibration, with the standard loop space construction. Due to the equivalence $\Omega^{2m-1}S^1 \simeq *$, we see $\Omega^{2m}S^{2n+1} \to \Omega^{2m}\mathbb{CP}^n$ is a weak equivalence. Simultaneously, the map $S^{2(n-m)+1} \to \Omega^{2m}S^{2n+1}$ which is adjoint via reduced suspension to the identity map $\Sigma^{2m}S^{2(n-m)+1} \to S^{2n+1}$ induces isomorphisms

$$\pi_i(S^{2(n-m)+1}) \otimes \mathbb{Q} \to \pi_i(\Omega^{2m}S^{2n+1}) \otimes \mathbb{Q} \cong \pi_{i+2m}(S^{2n+1}) \otimes \mathbb{Q}.$$

The result follows by Serre-Whitehead. □

We enumerate a particular basepoint $j_d^{m,n} \in \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ by

$$j_d^{m,n}([x_0 : \cdots : x_m]) \coloneqq [x_0^d : \cdots : x_m^d : 0 : \cdots : 0],$$

which we use in referring to two other results from the literature:

Lemma 4.2. *Fix $1 < m \leq n$ and $d \geq 1$. Then*

(a) (Sasao, [25]) Inclusion induces a rational homotopy equivalence

$$\mathrm{PW}_{d+1,n+1}(\mathbb{C}) \simeq \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) \simeq_{\mathbb{Q}} \mathrm{Map}_1(\mathbb{CP}^m, \mathbb{CP}^n).$$

(b) (Møller, [19]) The map $\theta_d^{m,n\prime} : \mathrm{Map}_{j_1^{m-1,n}} \to \mathrm{Map}_{j_d^{m-1,n}}$ given by

$$\theta_d^{m,n\prime}(g) \coloneqq g \circ j_d^{m,m}$$

induces multiplication by d^m on $\pi_{2(n-m)+1}(\mathrm{Map}_{j_1^{m-1,n}}) \to \pi_{2(n-m)+1}(\mathrm{Map}_{j_d^{m-1,n}})$.

Armed with these lemmas, we can prove the following:

Proposition 4.3. *Fix integers $1 < m \leq n$ and $d \geq 1$. Then the map*

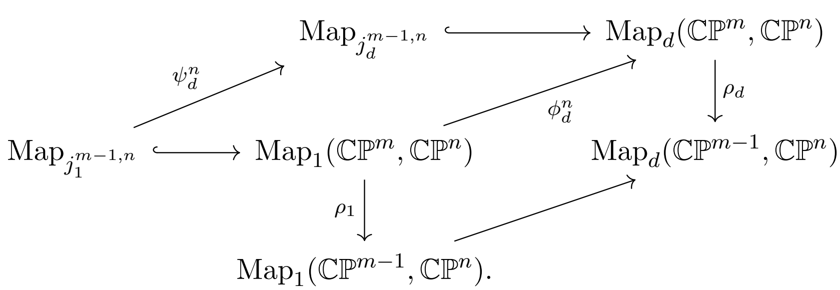
$$\theta_d^{m,n} : \mathrm{Map}_1(\mathbb{CP}^m, \mathbb{CP}^n) \to \mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n)$$

given by $\theta_d^{m,n}(g) \coloneqq g \circ j_d^{m,m}$ is a rational homotopy equivalence. Hence

$$H^*(\mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n); \mathbb{Q}) \cong \mathbb{Q}[y]/(y^{n-m+1}) \otimes \Lambda(x_{2(n-m)+3}, \cdots, x_{2n+1}), \tag{4.2}$$

where $|y| = 2$ and $|x_{2j+1}| = 2j+1$ for all $n - m < j \leq n$.

Proof. The argument is identical to that of 1.1 in Section 3, proceeding by induction on m:



(4.3)

Alternatively, one can use the long exact sequence in (rational) homotopy groups and the Five Lemma. Either method allows one to additionally recover a result due to Møller:

$$\pi_{2(n-m)+1}(\mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n)) \cong \mathbb{Z}/\tbinom{n+1}{m}d^m\mathbb{Z}.$$

□

Now we return to the holomorphic category by first considering the linear case. For any choice of $f \in \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)$, it is not hard to see that there is a homotopy equivalence

$$\mathrm{Hol}_f(\mathbb{C}) = \mathbb{C}^m \times (\mathbb{C}^{n-m+1} - 0) \simeq S^{2(n-m)+1}.$$

For example, the inclusion map $i : S^{2(n-m)+1} \to \mathrm{Hol}_{j_1^{m-1,n}}(\mathbb{C})$ given by

$$i(b_0, \ldots, b_{n-m})([z_0 : \cdots : z_m]) = [z_0 : \cdots : z_{m-1} : b_0 z_m : \cdots : b_{n-m} z_m] \tag{4.4}$$

is dual to a deformation retract $\mathrm{Hol}_{j_1^{m-1,n}}(\mathbb{C}) \to S^{2(n-m)+1}$. Hence the $E^{2(k+1)}$ transgression determines the Serre spectral sequence induced by the restriction map R, which is still a locally trivially fibration in the holomorphic category when $d = 1$:

$$\mathbb{Q} \cong H_{2(k+1)}(\mathrm{Hol}_{j_1^{m-1,n}}(\mathbb{C}); \mathbb{Q}) \xrightarrow{\partial} H_{2k+1}(S^{2k+1}; \mathbb{Q}) \cong \mathbb{Q}.$$

As in the proof of Proposition 3.1, this map is determined by the primary obstruction class as computed by Sasao [25]: the differential is $\partial(1) = \binom{n+1}{m}$.

We are now equipped to conclude this section by proving the remaining theorems.

Proof of Theorem 1.4. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) & \longrightarrow & \mathrm{Map}_d(\mathbb{CP}^m, \mathbb{CP}^n) \\ {\scriptstyle \psi_d^{m,n}} \uparrow & & \uparrow {\scriptstyle \theta_d^{m,n}} \\ \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) & \xrightarrow[\simeq_{\mathbb{Q}}]{} & \mathrm{Map}_1(\mathbb{CP}^m, \mathbb{CP}^n) \end{array}$$

Given Lemma 4.2 and Proposition 4.3, the composition from bottom left to top right corner is a rational equivalence. In light of Mostovoy's stability theorem [21], which shows that the top map is a homology equivalence in the range $i < d(2n-2m+1)-1$, the result follows. □

Having studied $\psi_d^{m,n}$ on rational homology, we can show the final two results:

Proof of Theorem 1.5 (a). As above, we consider bundle morphism (2.12) and the induced connecting morphisms on the resulting Serre spectral sequences. By Theorem 1.4, the map on fibers is a rational homotopy equivalence in the range $i < d - 1$ and we can conclude that the same is true of the map $\phi_d^{m,n} : \mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n) \to {}^m\mathrm{Hol}_d^{m,n}(\mathbb{C})$.

Because $\psi_d^{m,n}$ factors through $\phi_d^{m,n}$ and has a much larger isomorphism range, namely $i < d(2n - 2m + 1) - 1$, the result follows. Indeed, using this fact, one could make the additional statement that the inclusion

$$\sigma_d^{m,n} : {}^m\mathrm{Hol}_d^{m,n}(\mathbb{C}) \hookrightarrow \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$$

induces epimorphisms on rational homology through the range $i < d(2n - 2m + 1) - 1$. □

Proof of Corollary 1.6. We proceed via the Leray-Hirsch theorem. As the $\mathrm{PGL}_{m+1}(\mathbb{C})$-action is free and proper, the quotient map $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^m) \to \mathcal{X}_d^m$ is a fiber bundle. Hence the desired result amounts to proving that the inclusion of an orbit induces an epimorphism

$$H^i(\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^m); \mathbb{Q}) \to H^i(\mathrm{PGL}_{m+1}(\mathbb{C}); \mathbb{Q})$$

for all $i < d - 1$. This is precisely the content of Theorem 1.4. □

5 Arithmetic of algebraic maps

First we discuss the evidence, or lack thereof, for the maps in Theorems 1.4 and 1.5(a) to fail to be rational homotopy equivalences on the nose, using the machinery of étale cohomology introduced by Grothendieck and Deligne to prove the Weil conjectures.

The resultant polynomials used to define $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ are defined over $\mathbb{Z}$ and hence make sense over any field, in particular $\mathbb{F}_q$ and $\bar{\mathbb{F}}_q$. Counting the $\mathbb{F}_q$-points of these varieties is connected to topology via étale cohomology, together with the comparison/base-change theorems and Grothendieck-Lefschetz trace formula.

In short, if Y is a variety over $\mathbb{Z}$, one can reduce modulo an appropriate prime to obtain a variety over $\mathbb{F}_q$ just as easily as one can extend scalars. Hence one can associate to Y the étale cohomology $H^*_{\text{ét}}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell)$, where ℓ is prime to q, and at the same time associate the usual singular cohomology groups $H^*(Y(\mathbb{C}); \mathbb{Q})$. In many instances, in particular when Y is a smooth projective variety, one can also establish a natural isomorphism

$$H^i_{\text{ét}}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^i(Y(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}_\ell, \tag{5.1}$$

away from a finite (often empty) set of characteristics: see [6].

Recall that, for any variety Y defined over $\mathbb{F}_q$, we can define the Frobenius endomorphism

$$\mathrm{Frob}_q : Y(\bar{\mathbb{F}}_q) \to Y(\bar{\mathbb{F}}_q) \text{ via}$$
$$\mathrm{Frob}_q(x) = x^q.$$

We proceed using Hasse's fundamental observation, that the set $Y(\mathbb{F}_q)$ can be extracted from $Y(\bar{\mathbb{F}}_q)$ as the fixed points of the Frobenius map:

$$\#Y(\mathbb{F}_q) = \#\,\mathrm{Fix}\left(\mathrm{Frob}_q : Y(\bar{\mathbb{F}}_q) \to Y(\bar{\mathbb{F}}_q)\right).$$

In topology, the classical Lefschetz fixed point theorem is used to count fixed points of a continuous endomorphism $f : Y(\mathbb{C}) \to Y(\mathbb{C})$ in terms of the traces of the induced maps on singular homology. Fortunately, there is an analogous result for étale cohomology: the vector spaces $H^i_{\text{ét}}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ come as representations of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, wherein the eigenvalues of the induced Frobenius action on étale cohomology are known as weights. The key result we reference is the Grothendieck-Lefschetz trace formula [18], allowing us to compute the number $\#Y(\mathbb{F}_q)$ in terms of these weights:

$$\#Y(\mathbb{F}_q) = \sum_{i\geq 0}(-1)^i\,\mathrm{Trace}(\mathrm{Frob}_q : H^i_{\text{ét}}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell) \to H^i_{\text{ét}}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell)).$$

Unfortunately $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$ is not projective, so we cannot directly use the trace formula together with (5.1) to count points. To remedy this, note that Grothendieck-Lefschetz holds for any variety of finite type if we switch to compactly supported étale cohomology [6]. When Y is smooth we can apply Poincaré duality [18]:

$$H^i_{\text{ét},c}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^{2\dim Y - i}_{\text{ét}}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell(-\dim Y))^*$$

where * stands for the dual and $\mathbb{Q}_\ell(j)$ denotes a shift in Galois representations. Therefore for any smooth—but not necessarily projective—variety we have the modified formula:

$$\#Y(\mathbb{F}_q) = q^{\dim Y}\sum_{i\geq 0}(-1)^i\,\mathrm{Trace}(\mathrm{Frob}_q : H^i_{\text{ét},c}(Y_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell)^*). \tag{5.2}$$

The Hodge weights for the generators a_{2i-1} and c_i in the cohomology rings

$$H^i(\mathrm{PGL}_{m+1}(\mathbb{C});\mathbb{Q}) = \Lambda(a_3, \cdots, x_{2n+1}) \text{ and}$$

$$H^i(\mathrm{Gr}(\mathbb{CP}^m, \mathbb{CP}^n);\mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_{m+1}]/I,$$

where $|a_{2i-1}| = 2i-1$ and $|c_i| = 2i$, are known to be $-i$. These correspond to the counts

$$\#\,\mathrm{PGL}_{m+1}(\mathbb{F}_q) = \frac{(q^{m+1}-1)(q^{m+1}-q)\cdots(q^{m+1}-q^m)}{q-1} \text{ and}$$

$$\#\,\mathrm{Gr}(\mathbb{P}^m_{\mathbb{F}_q}, \mathbb{P}^n_{\mathbb{F}_q}) = \frac{(q^{n+1}-1)\cdots(q^{n+1-m}-1)}{(q^{m+1}-1)\cdots(q-1)}.$$

In light of the fiber bundle (2.1) we thus have the following count:

Lemma 5.1. *Fix integers $1 \le m \le n$. Then for any prime power q, we have*

$$\#\,\mathrm{Hol}_1(\mathbb{P}^m_{\mathbb{F}_q}, \mathbb{P}^n_{\mathbb{F}_q}) = \frac{\left(q^{n+1}-1\right)\cdots\left(q^{n+1}-q^m\right)}{q-1} \tag{5.3}$$

Assuming a sharp version of Theorem 1.4, we could compute $\#\,\mathrm{Hol}_d(\mathbb{P}^m_{\mathbb{F}_q}, \mathbb{P}^n_{\mathbb{F}_q})$ for $d > 1$ by appealing to the naturality of spectral sequences and algebraic maps with respect to Hodge weights. Under these conditions, we would arrive at the count

$$\#\,\mathrm{Hol}_d(\mathbb{P}^m_{\mathbb{F}_q}, \mathbb{P}^n_{\mathbb{F}_q}) = q^{(n+1)\binom{m+d}{d}} \frac{\left(1-q^{-(n+1)}\right)\cdots\left(1-q^{-(n-m+1)}\right)}{q-1} \tag{5.4}$$

simply by using codimensions of the subvariety

$$\psi_d^{m,n}(\mathrm{Hol}_1(\mathbb{CP}^m, \mathbb{CP}^n)) \subset \mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n)$$

to modify the leading power of q. Such a result would also correspond to a generalization of the count $\#(\mathrm{Rat}^n_d(\mathbb{F}_q)^*) = q^{d(n+1)}(1-q^{-n})$ established by Farb–Wolfson [8], in the form

$$\#\,\mathrm{Hol}_f(\mathbb{F}_q) = q^{(n+1)\binom{m+d-1}{d-1}}\left(1-q^{-(n-m+1)}\right) \tag{5.5}$$

for each choice of map $f \in \mathrm{Hol}_d(\mathbb{P}^{m-1}_{\mathbb{F}_q}, \mathbb{P}^n_{\mathbb{F}_q})$.

We have carried out point counts via SageMath of the associated spaces for small values of m, n, d and q (namely $m = n = 2$, $d \le 4$ and prime powers $q \le 11$) and arrived at exactly the counts predicted by (5.4). It should be noted that the cardinalities $\#\,\mathrm{Hol}_f(\mathbb{F}_q)$ are an auxiliary result of these computations and in all cases they have matched the prediction (5.5)—that is, they have not depended on f beyond the data of d, m, and n. This coincidence is surprising because it is not even clear when the restriction map $\mathrm{Hol}_d(\mathbb{CP}^m, \mathbb{CP}^n) \to \mathrm{Hol}_d(\mathbb{CP}^{m-1}, \mathbb{CP}^n)$ is a fiber bundle in the analytic topology.

Such numerical data does not prove the stronger result of rational homotopy equivalence even in this small extended range, even if they could be carried out for all prime powers q, but nonetheless the counts are encouraging.

We conclude with a final calculation: computing the number of non-degenerate algebraic morphisms $\mathbb{P}^1 \to \mathbb{P}^n$ of degree d defined over an arbitrary finite field $\mathbb{F}_q$. This can be thought of as a number-theoretic extension of Corollary 3.5, in light of the previous discussion.

Proof of Theorem 1.7. The spectral sequence (2.6) is reflected by the point count formula

$$\mathrm{Rat}_d^n(\mathbb{F}_q) = \sum_{r=1}^{n} {}_{\vartriangle}\mathrm{Rat}_d^r(\mathbb{F}_q) \binom{n+1}{r+1}_q, \tag{5.6}$$

where $\binom{n+1}{r+1}_q := \frac{(q^{n+1}-1)\cdots(q^{n+1-r}-1)}{(q^{r+1}-1)\cdots(q-1)}$ is the Gaussian binomial coefficient. With the count

$$\mathrm{Rat}_d^n(\mathbb{F}_q) = q^{(d-1)(n+1)} \frac{(q^n-1)(q^{n+1}-1)}{q-1} \tag{5.7}$$

and the base case ${}_{\vartriangle}\mathrm{Rat}_d^1(\mathbb{F}_q) = \mathrm{Rat}_d^1(\mathbb{F}_q) = q^{2(d-1)}(q^2-1)$, the claim follows by induction. □

Taking this point count as a prediction for the singular homology of the varieties ${}_{\vartriangle}\mathrm{Rat}_d^n(\mathbb{C})$, together with the use of Hodge weights, we indeed arrive at a conjecture which would extend Corollary 3.5. The associated spectral sequence (2.6), which would include many nonzero differentials but collapses at the E_2 page, correctly computes the cohomology calculated in Corollary 1.2. Moreover, in light of the work by Farb–Wolfson–Wood [9], the prediction can and should be expressed using the language of homological density:

Conjecture 5.2 (Spherical homological density)**.** *Fix integers* $1 < n < d$. *Then*

$$\frac{P_t({}_{\vartriangle}\mathrm{Rat}_d^n(\mathbb{C}))}{P_t(\mathrm{Rat}_d^n(\mathbb{C}))} = \prod_{i=1}^{n-1}(1+t^{2(d-i)-1}). \tag{5.8}$$

Note that, if true, the same result would hold when stated with based mapping spaces, where rational functions correspond to configurations of so-called colored points. Such a statement could therefore be interpreted as a homological density statement regarding the subspace of configurations which produce non-degenerate maps, predicted by arithmetic.

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