THESIS SUMMARY

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1. Introduction

The main goal of my thesis is the application of logical and computability-theoretic techniques to better understand the foundational nature of structures and theorems from different branches of mathematics. To achieve this goal, I examined the effective (i.e. computable) content of structures and theorems from several different branches of math, including computability theory and model theory (Chapter 1), fractal dimension and probability theory (Chapter 2), commutative and noncommutative rings (Chapters 3 and 4), and measure theory (Chapter 5). My thesis contributes to these areas of mathematics by examining the effective content of central structures and/or theorems in each of them, and in so doing sheds light on their fundamental (i.e. logical) nature. The next five sections describe the five chapters of my thesis in sequence.

2. Classifying model-theoretic properties

Chapter 1 of my thesis deals with computability theory and computable model theory. This work grew out of an effective analysis of prime models, and extends previous work of Morley, Harrington, Millar, and others.

Definition 2.1. Let $\mathcal{L}$ be a language, and T a theory of $\mathcal{L}$. A model $\mathcal{M}$ of T is *prime* if $\mathcal{M}$ can be elementarily embedded into any other model of T. If T is complete and $\mathcal{L}$ is countable, then it is known that there is a unique prime model of T up to isomorphism.

Example 2.2. Let T be the theory of algebraically closed fields of characteristic zero. Then any prime model of T is isomorphic to the field of algebraic numbers.

Recently, Csima, Hirshfeldt, Knight, and Soare [CHKS04] studied the *prime bounding* Turing degrees.

Definition 2.3. Let $\mathcal{L}$ be a computable language, and T a complete theory of $\mathcal{L}$.

(1) We say that T is *decidable* if there is an algorithm that can decide whether or not T proves any given $\mathcal{L}$-sentence.
(2) We say that T is *atomic* if for every formula $\varphi(\overline{x})$ consistent with T, there is a principle type containing $\varphi(\overline{x})$.

It is known that for any given complete atomic theory T, there is a prime model of T.

Definition 2.4. Let $\mathbf{d}$ be a Turing degree (of unsolvability). Then $\mathbf{d}$ is *prime bounding* if it can compute the elementary diagram of *any* complete atomic decidable (CAD) theory T.

In [CHKS04], the authors introduce nine (Turing) degree invariant predicates of a set $A \subseteq \mathbb{N}$ and show that they are equivalent when restricted to sets that are computable in Turing's halting set K. The nine predicates come from different branches of mathematics, and talk about different types of mathematical structures including p-groups, equivalence structures, prime models, Baire category (i.e. forcing) constructions, and nonlow$_2$ Turing degrees. Chapter 1 of my thesis sheds new light on these structures by expanding on the results of [CHKS04]. In particular, I determined all logical implications between the nine predicates in general, and showed that, for Turing degrees that are not computable in K, the predicates fall into three distinct equivalence classes under logical implication. There are mainly two types of results that I prove in Chapter 1. The first type extend the results of [CHKS] by showing (via new and different proofs) that equivalences which [CHKS04] proved hold for sets computable in K can be extended to the Turing degrees in general. The second type of result says that two predicates that are equivalent for sets computable in Turing's K (and are therefore similar from a certain point of view) are not equivalent when extended to the Turing degrees in general. The proofs contained in Chapter 1 contrast with those of [CHKS] who studied a more restrictive case and therefore had more tools at their disposal, such as the Shoenfeld limit lemma. One reason why Chapter 1 is interesting and significant is because it provides new characterizations of a class of Turing degrees that can perform a type of effective forcing construction first introduced by Shinoda and Slaman in [SS00]. In particular I proved that these Turing degrees are equal to the prime bounding degrees. More generally, however, I was able to compare the logical nature of various different mathematical structures in the most general context. My results also contrast with some recent results of Hirschfeldt, Shore, and Slaman [HSS09], who examined some the nine predicates in the context of reverse mathematics. Chapter 1 has been published in the *Journal of Symbolic Logic* [Con08a].

3. Effective packing dimension of Π^0_1-classes

Chapter 2 deals with effective fractal dimension. The first notion of fractal dimension was introduced by Hausdorff in 1919 [Hau19] as a tool for studying the geometry of fractals, to which calculus cannot be applied. Later on, mathematicians developed other notions of fractal dimension, including packing dimension which is dual to Hausdorff dimension, for the same purpose. Recently, Jack Lutz [Lut03] effectivized the notions of Hausdorff dimension and packing dimension by characterizing them in terms of martingales (i.e. betting strategies) from probability theory, and then effectivizing the associated martingales.

Definition 3.1. Let $2^{<\omega}$ denote the set of finite binary sequences. Let 2^{ω} denote the set of infinite binary sequences. Let $s \in \mathbb{R}^{\geq 0}$. A function $d : 2^{<\omega} \rightarrow \mathbb{R}^{\geq 0}$ is called an *s-gale* if it satisfies, for every $\sigma \in 2^{<\omega}$, the following equality

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2^s}.$$

Example 3.2. A martingale is just a 1-gale from Definition 3.1 above. Let $d : 2^{<\omega} \rightarrow \mathbb{R}^{\geq 0}$ be a martingale. Then we have that for every $\sigma \in 2^{<\omega}$,

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$

In other words, for every finite binary sequence $\sigma \in 2^{<\omega}$, we have that $d(\sigma) \in \mathbb{R}^{\geq 0}$ is the average of $d(\sigma 0)$ and $d(\sigma 1)$.

We think of d as a strategy for betting on infinite binary sequences $f \in 2^{\omega}$. For any given $f \in 2^{<\omega}$, and $\sigma \subset f$, $d(\sigma)$ is the amount of capital that one would have after placing $|\sigma|$-many bets (here $|\sigma| \in \mathbb{N}$ denotes the number of bits in $\sigma \in 2^{<\omega}$). We think of $s \in \mathbb{R}^{\geq 0}$ as a "fairness factor", because for larger values of s the betting is less fair.

Definition 3.3. Let $s \in \mathbb{R}^{\geq 0}$, and $d : 2^{<\omega} \rightarrow \mathbb{R}^{\geq 0}$ be an s-gale. The *strong success set* of d is the set of $f \in 2^{\omega}$ such that $\lim_n d(f_n) = \infty$, where $n \in \mathbb{N}$ and $f_n \in 2^{<\omega}$ denotes the first n bits of f. In other words, the strong success set of d is the set of infinite binary sequences on which d makes (banks) arbitrarily large amounts of capital.

The following theorem characterizes classical packing dimension in terms of s-gales.

Theorem 3.4. [AHLM07] *Let $X \subseteq 2^{\omega}$, then the (classical) packing dimension of X is the infemum of the set of $s \in \mathbb{R}^{\geq 0}$ such that there is an s-gale $d : 2^{<\omega} \rightarrow \mathbb{R}^{\geq 0}$ for which X is contained in the strong success set of d.*

To obtain Lutz's notion of *effective* packing dimension, one simply requires the s-gales of Theorem 3.4 above to be computably approximable from below. The case of Hausdorff dimension is the same, except that we replace $\lim_n$ by $\limsup_n$ in the definition of success set (in other words, we only require that d makes arbitrarily large amounts of capital infinitely often on $f \in 2^{\omega}$). It follows that the effective Hausdorff (packing) dimension of a set $X \subseteq 2^{\omega}$ is always greater than or equal to the classical Hausdorff (packing) dimension of X. However, these two notions are different in general.

After developing the theory of effective fractal dimension, Lutz and others related effective fractal dimension to central ideas in computability theory, such as algorithmic randomness and Kolmogorov complexity.

Definition 3.5. Let Φ be a prefix-free universal Turing machine (i.e. a universal Turing machine with prefix-free domain), and let $\sigma \in 2^{<\omega}$. We define the *Kolmogorov complexity* of σ with respect to Φ, denoted by $K(\sigma)$, to be the length of the shortest $\tau \in 2^{<\omega}$ such that Φ halts on input τ and outputs σ. It is known that Kolmogorov complexity is well-defined up to a constant that depends only on Φ (and not on σ or τ).

Definition 3.6. An infinite binary sequence $f \in 2^\omega$ is called *random* if there is a constant $c \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, the Kolmogorov complexity of the first n bits of f is greater than or equal to $n - c$. It is easy to see that the set of random sequences does not depend upon our choice of universal prefix-free machine in Definition 3.5 above.

Next, we state a nice result of Mayordomo, Lutz, and others that characterizes effective Hausdorff (packing) dimension in terms of Kolmogorov complexity.

Theorem 3.7. [Lut03, AHLM07] *Let $f \in 2^\omega$ be an infinite binary sequence, and let f_n denote the first n bits of f. Let $X \subseteq 2^\omega$. Then:*

(1) *The effective Hausdorff dimension of $f \in 2^\omega$ is given by*

$$\liminf_n \frac{K(f_n)}{n}.$$

(2) *The effective packing dimension of $f \in 2^\omega$ is given by*

$$\limsup_n \frac{K(f_n)}{n}.$$

(3) *The effective Hausdorff (packing) dimension of $X \subseteq 2^\omega$ is the supremum of the effective Hausdorff (packing) dimension of its elements.*

Example 3.8. By definition, if $f \in 2^\omega$ is random, then f has effective Hausdorff dimension 1, and effective packing dimension 1. If $g \in 2^\omega$ is computable, then it follows that g has effective Hausdorff dimension 0 and effective packing dimension 0. If $f \in 2^\omega$ is random and $g \in 2^\omega$ is computable, then it follows that $f \oplus g \in 2^\omega$ has effective Hausdorff dimension $\frac{1}{2}$ and effective packing dimension $\frac{1}{2}$. In this sense, effective fractal dimension gives a way of defining "partial randomness."

After developing the theory of effective fractal dimension, Lutz asked for a *correspondence principle* between classical Hausdorff dimension (as defined by Hausdorff) and effective Hausdorff dimension. More explicitly, Lutz was asking for a class of sets whose members have equal classical and effective Hausdorff dimensions (it is known that the classical fractal dimension of a set X is always bounded above by the effective fractal dimension of X, but these notions are different in general). Essentially, Lutz wanted to know whether his new effective version of dimension is equivalent to the classical versions in a sufficiently simple, but nontrivial context.

Definition 3.9. The standard topology on 2^ω (i.e. the set of infinite binary strings) is generated by basic clopen sets of the form $[\sigma]$, $\sigma \in 2^{<\omega}$, where $[\sigma] \subseteq 2^\omega$ consists of the elements of 2^ω that have σ as an initial segment.

For any given set $A \subseteq 2^{<\omega}$, we let $[A] \subseteq 2^\omega$ consist of the set of infinite binary strings that extend some element of A.

A set $U \subseteq 2^\omega$ is said to be *effectively open* if there is a computably enumerable set of finite binary strings, $A \subseteq 2^{<\omega}$, such that $U = [A]$. A subset of 2^ω is said to be *effectively closed* if its complement is effectively open. By definition, every effectively open set is open and every effectively closed set is closed.

A correspondence principle for Hausdorff dimension was discovered by John Hitchcock [Hit05], who showed that unions of effectively closed sets have their effective fractal dimension equal to their classical fractal dimension. Later, Lutz asked for a correspondence principle for packing dimension. Chapter 2 of my thesis essentially shows that, contrary to the case of Hausdorff dimension, there is *no* correspondence principle for packing dimension. In particular, I constructed an effectively closed set with classical packing dimension 0, and effective packing dimension 1. This result is interesting, because it highlights differences between Hausdorff dimension and its dual packing dimension. Essentially, this difference boils down to genericity, and my proof essentially shows that there are sufficiently generic but simple mathematical objects that have high packing dimension, but low Hausdorff dimension. Moreover, the techniques used in Chapter 2 have been of recent interest to computability theorists because they provide a new systematic method for constructing sets of high effective packing dimension. These techniques have since been employed by computability theorists to construct sets that have high packing dimension and satisfy other properties. The results of Chapter 2 have appeared in the *Proceedings of the American Mathematical Society* [Con08b].

4. Chain conditions in computable rings

In Chapter 3 I studied effective algebra and the reverse mathematics of ring theory. This work extends previous work of computable ring theorists in examining ideal membership problems and algorithms. Ideal membership algorithms have numerous applications in mathematics and in industry, making them among the most studied mathematical objects in all of algebra. Effective ring theory has its roots in the work of Kronecker [Kro82], van der Waerden [vdW], and other algebraists who examined ideal membership problems in the guise of splitting algorithms for polynomials over a field. Throughout this section *ring* will mean a commutative ring with an identity element.

Definition 4.1. A *computable ring* is a subset of natural numbers endowed with the structure of a ring such that the operations of addition, subtraction, and multiplication are given by computable functions.

More recently, Baur [Bau74] showed that computable Artinian and Noetherian rings have decidable ideal membership problems. In the Artinian case Baur was able to construct a uniform algorithm that computes every ideal (given a finite set of generators), while in the Noetherian case he proved that no such uniform algorithm exists. In Chapter 3 of my thesis, I attempted to classify the reverse mathematical strength of the theorem from classical commutative algebra that says "Every Artinian ring is Noetherian." In other words, this theorem says that any ring that has an infinite strictly ascending chain of ideals must also contain an infinite strictly descending sequence of ideals. Essentially, I determined the Turing strength required to compute an infinite strictly descending chain of ideals in a ring with a uniformly computable strictly increasing chain of ideals. I showed that, not only is this problem undecidable, but if a Turing degree can accomplish this then it must also compute a complete and consistent extension of Peano arithmetic. This work is related to recent work of Downey, Lempp, and Mileti [DLM07], and involves techniques that blend algebra with computability theory. More precisely, the construction of a computable ring that contains a uniformly computable strictly ascending chain of ideals, but such that every infinite strictly descending chain of ideals in the ring codes a complete and consistent extension of Peano arithmetic must be somewhat subtle because such a ring must contain arbitrarily large computable strictly descending chains of ideals (but no infinite ones). These results are significant because they imply that solving the ideal membership problem for ascending chains of ideals can be quite different from solving the ideal membership problem for descending chains of ideals, even in the *same* ring. Chapter 3 has been accepted for publication in the *Transactions of the American Mathematical Society* [Conb].

5. The complexity of radicals in noncommutative rings

Chapter 4 of my thesis classifies the complexity of the prime radical and Levitzki radical in noncommutative rings (with identity). Throughout this section, by *ring* we will mean a (possibly) noncommutative ring with identity. Again, this work builds on recent results of Downey, Lempp, and Mileti [DLM07], who determined the complexity of the nilradical and Jacobson radical.

Definition 5.1. A formula $\varphi(\overline{x})$ is Σ^0_0 (or, equivalently, Π^0_0) if it is a computable formula with no quantifiers. Let $n \in \mathbb{N}$, $n > 0$. A formula is Σ^0_n if it is of the form $(\exists n)\varphi(n, \overline{x})$ and $\varphi(n, \overline{x})$ is Π^0_{n-1}. A formula is Π^0_n if it is of the form $(\forall n)\varphi(n, \overline{x})$ and $\varphi(n, \overline{x})$ is Σ^0_{n-1}. A formula is *arithmetic* if it is Σ^0_n or Π^0_n, for some $n \in \mathbb{N}$. We say that a set of tuples of natural numbers is Σ^0_n (Π^0_n, arithmetic) if it can be defined by a Σ^0_n (Π^0_n, arithmetic) formula.

Example 5.2. Let $e, n, s \in \mathbb{N}$, and let $\varphi(e, n, s)$ be the computable formula with no quantifiers which says that the e-th Turing machine halts on input n after s-many steps. Then $\varphi(e, n, s)$ is a Π^0_0 formula. Therefore, the formula $(\exists s)[\varphi(e, n, s)$ halts after s-many steps] is Σ^0_1. Hence, the set of pairs $\langle e, s\rangle$ that satisfy the formula of the last sentence is Σ^0_1. The formula $(\forall s)[\varphi(e, n, s)$ halts after s-many steps] is Π^0_1. The formula $(\forall n)(\exists s)[\varphi(e, n, s)$ halts after s-many steps] is Π^0_2.

Definition 5.3. A formula $\varphi(\overline{x})$ is Π^1_1 if it is of the form $(\forall A)\psi(A, \overline{x})$, where A ranges over subsets of natural numbers, and $\psi(\overline{x})$ is arithmetic. Π^1_1 formulas are more complicated than arithmetic formulas, because they permit quantification over set/function variables rather than individual natural numbers. A set of tuples of natural numbers is Π^1_1 if it has a Π^1_1 definition.

Definition 5.4. Let Γ be a complexity class (such as Σ_n^0, Π_n^0, or Π_1^1). We say that $A \subseteq \mathbb{N}$ is Γ-*complete* if A belongs to Γ and every other set belonging to Γ can be computably reduced to A. In other words, A is among the most complicated sets belonging to class Γ.

We now recall a well-known definition and theorem from ring theory.

Definition 5.5. Let R be a ring. The *Jacobson radical* of R, $M \subset R$, is the intersection of all (two-sided) maximal ideals of R. Note that if R is a computable ring, then our definition of the Jacobson radical of R is Π_1^1, since it quantifies over subsets (ideals) of R.

Theorem 5.6. *Let R be a ring, and $M \subset R$ be the Jacobson radical of R. Then, for every $x \in R$, we have that $x \in M$ if and only if*

$$(\forall r, s \in R)(\exists u \in R)[(1_R - rxs)u = 1_R].$$

In other words, $x \in R$ is in the Jacobson radical of R if and only if for all $r, s \in R$ we have that $1_R - rxs$ is a unit in R.

Note that, if R is a computable ring, then Theorem 5.6 above says that the Jacobson radical of R, $M \subset R$, is a Π_2^0 set (since it characterizes M by a formula of the form $\forall\exists\cdots$). On the other hand, the definition of M is Π_1^1, since it quantifies over maximal ideals of R.

The main result of Chapter 4 says that there is a computable noncommutative ring whose prime radical (i.e. the intersection of all prime ideals) is Π_1^1-complete. This significantly contrasts with the cases of nilradical, Jacobson radical, and Levitzki radical, which are Σ_1^0-complete, Π_2^0-complete, and Π_2^0-complete, respectively. Elegantly stated, the main result of Chapter 4 says that *any* definition of the prime radical of a noncommutative ring *must* have a universal quantifier ranging over subsets of the ring. This is a significant and interesting result because it contrasts with the case of Jacobson radical (i.e. the intersection of all maximal ideals), where a simpler (Π_2^0) definition is possible, and the case of commutative rings, where the prime radical is equal to the nilradical and is therefore at most Σ_1^0-complete. Furthermore, this result highlights major differences between the behavior of prime and maximal ideals in noncommutative rings. Chapter 4 has been accepted for publication in the *Journal of Algebra* [Conc].

6. Effectively approximating Borel sets by open sets

Chapter 5 of my thesis resolves a recent outstanding question in effective measure theory, and is related to recent results of J. Miller and others in the field of algorithmic randomness and Kolmogorov complexity. Recently, there has been much interest in algorithmic randomness and Kolmogorov complexity (Definition 3.5) by computability theorists, and many connections between randomness and computability have been established. One such connection involves relating the random-theoretic properties of infinite binary sequences to their ability to effectively approximate Borel sets with respect to Lebesgue measure. Such results were first obtained by Kurtz [Kur] and Kautz [Kau] over twenty years ago, and more recently by Kjos-Hanssen, Miller, Solomon [KH07, KHMS], and others.

Definition 6.1. The Lebesgue measure on 2^ω is the unique measure obtained by setting the measure of the basic clopen set $[\sigma]$, $\sigma \in 2^{<\omega}$, to be $2^{-|\sigma|}$. Here $[\sigma] \subseteq 2^\omega$ is the set of elements of 2^ω that extend $\sigma \in 2^{<\omega}$, and $|\sigma|$ denotes the length of $\sigma \in 2^{<\omega}$.

Definition 6.2. Let $X \subseteq 2^\omega$. We say that X is a Σ_1^0-class if X is effectively open; we say that X is a Π_1^0-class if X is effectively closed (see Definition 3.9 above). For $n \in \mathbb{N}$, $n > 1$, we say that X is a Σ_n^0-class if X is the union of an effective (i.e. computable) sequence of Π_{n-1}^0-classes; we say that X is a Π_n^0-class if X is the intersection of an effective (i.e. computable) sequence of Σ_{n-1}^0-classes. Equivalently, Σ_n^0-classes and Π_n^0-classes are those subsets of 2^ω that can be defined by Σ_n^0 and Π_n^0 formulas (see Definition 5.1), respectively.

Example 6.3. Σ_2^0-classes are effective unions of effectively closed sets, and are therefore F_σ. On the other hand, Π_2^0-classes are effective intersections of effectively open sets, and are therefore G_δ.

Recently, Bienvenu, Muchnik, Shen, and Vereshchagin [BMSV] examined a theorem of J. Miller [Mil04] and others in the context of effective measure theory, and related it to notions of effective approximations of Borel sets in measure. After proving a special case of a more general theorem, [BMSV] asked whether or not the general theorem also holds. Recently, I showed that the more general theorem is also true. More specifically, I proved that Turing's halting set can uniformly approximate the liminf of a uniformly effectively open sequence of sets via an open set arbitrarily close in measure. The liminf of a sequence of sets is the set of points contained in all but finitely many members of the sequence. This result is significant, because the arithmetic complexity of the liminf of a uniformly effectively open sequence of sets is in general Σ^0_3, but open sets relative to the halting set are Σ^0_2. Therefore, my result says that one can uniformly approximate (more complicated) Σ^0_3-classes in measure by (simpler) Σ^0_2-classes. I also show that one cannot do better than Σ^0_2 in this context. Chapter 5 has been submitted for publication [Cona].

7. Summary

Each of the five chapters of my thesis sheds new light on foundational aspects of a different branch of mathematics. Chapter 1 classifies the general effective content of properties arising from several different branches of mathematics, and grew out of the work of model theorists such as Morley and Harrington. Chapter 2 deals with fractal dimension, and relates logical notions such as genericity to probabilistic notions such as martingales, and geometric notions such as fractal dimension introduced by Hausdorff almost 100 years ago. Chapter 3 gives new insights into ideal membership problems in computable rings first studied by Kronecker in the 1880s, while Chapter 4 highlights significant differences between prime and maximal ideals in noncommutative rings. Meanwhile, Chapter 5 solves a recent problem in effective measure theory, determining the effective complexity required to approximate Borel sets by open sets with respect to Lebesgue measure. In the future I will use the tools of mathematical logic and computability theory to provide an even deeper analysis of an even broader spectrum of mathematics.

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