HOMEWORK 5 — SURFACES AND FUNDAMENTAL GROUPS

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This homework is due Wednesday November 22nd at the start of class. Remember that the notation

$$\langle e_1, e_2, \ldots, e_n | w_1, w_2, \ldots, w_m \rangle$$

denotes the group whose generators are equivalence classes of words in the generators e_i and e_i^{-1}, where two words are equivalent if one can be obtained from the other by inserting or deleting $e_i e_i^{-1}$ or $e_i^{-1} e_i$ or by inserting or deleting a word w_j or its inverse.

Problem 1. Let S_n be a surface obtained from a regular $4n$–gon by identifying opposite sides by a translation. What is the Euler characteristic of S_n? What surface is it? Find a presentation for its fundamental group.

Problem 2. Let S be the surface obtained from a hexagon by identifying opposite faces by translation. Then the fundamental group of S should be given by the generators a, b, c corresponding to the three pairs of edges, with the relation $abca^{-1}b^{-1}c^{-1}$ coming from the boundary of the polygonal region.

What is wrong with this argument? Write down an actual presentation for $\pi_1(S, p)$. What surface is S?

Problem 3. The *four–color theorem* of Appel and Haken says that any map in the plane can be colored with at most four distinct colors so that two regions which share a common boundary segment have distinct colors. Find a cell decomposition of the torus into 7 polygons such that each two polygons share at least one side in common.

Remark. It turns out that only 7 colors are required for any map on the torus. In general, for a genus g surface, the number of necessary colors is the smallest integer greater than or equal to $\frac{7+\sqrt{1+48g}}{2}$.

Problem 4. Define the *connect sum* of two surfaces S_1, S_2 to be a new surface obtained by removing a small disk from each of S_1, S_2 and sewing the two boundary components together. The surface so obtained is denoted $S_1 \# S_2$. Show for orientable S_1, S_2 that this operation is well–defined. That is, the resulting surface does not depend on which disks are removed from S_1, S_2 or how the boundaries are glued. What is the formula for $\chi(S_1 \# S_2)$ in terms of $\chi(S_1)$ and $\chi(S_2)$?

Problem 5. If F, G are two orientable surfaces, show that F is a covering space of G of degree n if and only if

$$\chi(F) = n\chi(G)$$

In particular, there exist degree n covers by the torus of itself of arbitrary degree.

Problem 6. Show that every nonorientable surface S has a degree 2 orientable cover. Hint: show that there is a homomorphism $\pi_1(S, p) \to \mathbb{Z}/2\mathbb{Z}$ whose kernel consists of the loops which are orientation–preserving.

Problem 7. The *Klein bottle* K is the surface obtained from a square by gluing top and bottom sides by translation, and gluing left and right sides with a "twist". Show that a presentation for the fundamental group is given by

$$\pi_1(K,p) \cong \langle a,b | aba^{-1}b \rangle$$

Find another cell decomposition of K with one polygon and one vertex that gives another presentation for the fundamental group as

$$\pi_1(K,p) \cong \langle a,b | a^2b^2 \rangle$$

Problem 8. What orientable surfaces without boundary can be built from heptagons, where 3 of these are required to meet around every vertex? What surfaces can be built from triangles, where 7 of these are required to meet around every vertex?

Note: a *heptagon* is a 7–sided polygon.

Problem 9. A *pair of pants* is another word for a disk with two holes (can you see why?). Show that a surface of genus g can be decomposed into $2g-2$ pairs of pants. Show that the number of such decompositions, up to combinatorial equivalence, is equal to the number of graphs with $2g-2$ vertices, and 3 edges at every vertex. Enumerate the number of such graphs for $g \le 5$.

Problem 10 (hard). Recall the notation $[a,b] = aba^{-1}b^{-1}$ for elements a,b in a group.

The *abelianization* of a group G is an abelian group A and a homomorphism $\varphi : G \to A$ such that if B is any abelian group, and $\phi : G \to B$ is any homomorphism, there is a unique homomorphism $\psi : A \to B$ (which might depend on ϕ) such that $\psi\varphi = \phi$.

- Show that if A exists, then it is unique in the sense that for any other $\varphi' : G \to A'$ with the properties above, there is an isomorphism $\rho : A' \to A$ such that $\rho\varphi' = \varphi$.
- If $G = \langle e_1, e_2, \ldots, e_n | w_1, w_2, \ldots, w_m \rangle$ is a finitely presented group, show
$$A = \langle e_1, e_2, \ldots, e_n | w_1, \ldots, w_m, [e_1,e_2], \ldots, [e_i,e_j], \ldots, [e_{n-1}, e_n] \rangle$$
is a presentation of the abelianization of G, where the homomorphism $\varphi : G \to A$ sends the equivalence class of w in G to the equivalence class of w in A for each word $w \in G$.
- Calculate presentations for the abelianization of the fundamental groups of the Klein bottle and of the genus 2 surface. Identify these abelian groups explicitly.

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