NOTES ON RATIONAL HOMOTOPY THEORY

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Abstract. This is a very sketchy summary of some of the material in [2], Chapters 8–11. There is no pretension to originality, and I have not even attempted to reorganize the material in a meaningful way. The main purpose of writing these notes was to digest and to verify the claims in that section to my satisfaction. Since [2] is long out of print and in any case poorly typeset, these notes may possibly be of value to someone else.

1. Postnikov towers

Definition 1.1. Let X be a CW complex. A *Postnikov tower for* X is a sequence of spaces $\cdots \to X_n \to \cdots \to X_2 \to X_1$ together with compatible maps $f_i : X \to X_i$ such that

(1) for each i, the homotopy groups of X_i vanish in dimension $> i$; and
(2) each map $f_i : X \to X_i$ induces an isomorphism in homotopy groups in dimensions $\le i$.

If we replace each $X_{n+1} \to X_n$ by a fibration up to homotopy, then the fiber is a $K(\pi_{n+1}, n+1)$. A space X with the homotopy type of a CW complex is *abelian* if $\pi_1(X)$ is abelian, and acts trivially by conjugation on $\pi_*(X)$. If X is abelian, each fibration $K(\pi_{n+1}, n+1) \to X_{n+1} \to X_n$ can be extended to a fibration $X_{n+1} \to X_n \to K(\pi_{n+1}, n+2)$ which is therefore classified by the map $k_n : X_n \to K(\pi_{n+1}, n+2)$, or equivalently, by $[k_n] \in H^{n+2}(X_n; \pi_{n+1})$. Such an X can be recovered (up to homotopy) from the groups π_n and the k-invariants k_n. See e.g. [3], pp. 410–414.

Example 1.2. Suppose X has dimension $\le n$, and suppose we know X_n. Since $X \to X_n$ induces isomorphisms in π_i for $i \le n$ it induces isomorphisms in H_i for $i \le n$, so by relative Hurewicz theorem (applied to homology), it is surjective on H_{n+1}. By assumption on X, we conclude $H_{n+1}(X_n) = 0$. From the les in homotopy groups of a fibration $F \to X \to X_n$, plus the Hurewicz theorem, $\pi_{n+1} = H_{n+2}(X_n)$, and $k \in H^{n+2}(X_n; \pi_{n+1})$ is the identity morphism. Thus, the homotopy type of X can be (inductively) recovered "formally" from X_n.

2. Localization

Definition 2.1. Let $\mathcal{P}$ be a set of primes in $\mathbb{Z}$, possibly empty, and let $\mathbb{Z}_\mathcal{P}$ be the subring of $\mathbb{Q}$ obtained by inverting all primes not in $\mathcal{P}$. If A is an abelian group, let $A \to A \otimes \mathbb{Z}_\mathcal{P}$ be the localization of A.

Lemma 2.2. *The $\mathbb{Z}$-module structure on A extends to a $\mathbb{Z}_\mathcal{P}$-module structure if and only if the localization map $A \to A \otimes \mathbb{Z}_\mathcal{P}$ is an isomorphism; equivalently, if*

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and only if the multiplication map $\times\ell : A \to A$ is an isomorphism for all primes ℓ not in $\mathcal{P}$.

Definition 2.3. A connected abelian space X is called *$\mathcal{P}$-local* if $\pi_i(X)$ is a $\mathbb{Z}_\mathcal{P}$-module for all i. A map $X \to X'$ of abelian spaces is a *$\mathcal{P}$-localization of* X if X' is $\mathcal{P}$-local, and the map induces isomorphisms $\pi_*(X) \otimes \mathbb{Z}_\mathcal{P} \to \pi_*(X') \otimes \mathbb{Z}_P = \pi_*(X')$.

Theorem 2.4 (Localization theorem)**.** *Let $\mathcal{P}$ be a set of primes.*

1. *For every abelian space X there exists a $\mathcal{P}$-localization $X \to X'$.*
2. *A map $X \to X'$ of abelian spaces is a $\mathcal{P}$-localization iff $\widetilde{H}_*(X')$ is a $\mathbb{Z}_\mathcal{P}$-module, and the map induces isomorphisms*
$$\widetilde{H}_*(X) \otimes \mathbb{Z}_\mathcal{P} \to \widetilde{H}_*(X') \otimes \mathbb{Z}_P = \widetilde{H}_*(X')$$
3. *$\mathcal{P}$-localization is a functor of homotopy types.*

See [4], pp. 35–38 for a proof. Note that X' has the homotopy type of a CW complex by our convention about abelian spaces.

In the sequel, we are exclusively interested in the case that $\mathcal{P}$ is empty, in which case $\mathbb{Z}_\mathcal{P} = \mathbb{Q}$, and we denote the $\mathbb{Q}$-localization by $X_\mathbb{Q}$.

3. Differential graded algebras

3.1. **Definition.**

Definition 3.1. A *differential graded algebra* $A^* = \oplus_{p\ge 0} A^p$ is a graded vector space over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ together with maps

1. a differential $d : A^* \to A^{*+1}$ with $d^2 = 0$; and
2. a product $A^p \otimes A^q \to A^{p+q}$ satisfying $\alpha\beta = (-1)^{pq}\beta\alpha$

for which the differential satisfies the Leibniz rule:
$$d(\alpha\beta) = d\alpha\beta + (-1)^p \alpha d\beta$$

Hereafter we abbreviate "differential graded algebra" to dga.

A morphism $\rho : A^* \to B^*$ of dgas (over the same field) is a grading-preserving linear map of vector spaces which is an algebra morphism (i.e. $\rho(\alpha\beta) = \rho(\alpha)\rho(\beta)$) and respects d (i.e. $d\rho(\alpha) = \rho(d\alpha)$).

Example 3.2. If M is a smooth manifold, the de Rham complex $\Omega^*(M)$ is a dga over $\mathbb{R}$ with exterior d as differential, and wedge $\wedge$ as product. If X is any space, $H^*(X;\mathbb{Q})$ is a dga over $\mathbb{Q}$ with $d = 0$ and cup product $\smile$ as product. However, the singular cochain complex $C^*(X;\mathbb{Q})$ is *not* a dga, since cup product of cochains is not (skew)-commutative.

3.2. **PL de Rham complex.**

The most important example for our applications is the *PL de Rham complex* $A^*_\mathbb{Q}(K)$, which we shortly define.

Let K be a simplicial complex, i.e. a union of n-simplices, each with a natural parameterization by the standard n-simplex $\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1\}$.

Definition 3.3. A *rational form* on Δ^n is the restriction to Δ^n of a form $\sum \varphi_I dt_I$ (taken over multi-indices I) where the φ_I are polynomials with $\mathbb{Q}$ coefficients.

Let $A^*(\Delta^n)$ denote the algebra of rational forms on Δ^n. This is a dga, with respect to ordinary exterior d and wedge product. Note that this algebra satisfies relations $\sum t_i = 1$ and as a consequence, $\sum dt_i = 0$.

For each face inclusion $\Delta^k \to \Delta^n$ there is a restriction map $A^*(\Delta^n) \to A^*(\Delta^k)$ which is a surjective morphism of dgas.

Definition 3.4. Let K be a simplicial complex. The PL de Rham complex $A^*_{\mathbb{Q}}(K)$ (or just $A^*(K)$ if $\mathbb{Q}$ is understood) is the set of collections of forms (ω_σ), one for each simplex $\sigma \in K$, so that $\omega_\sigma \in A^*(\sigma)$ for each σ, and if τ is a face of σ, then $\omega_\sigma|\tau = \omega_\tau$.

Since face restriction maps are morphisms of dgas, the operations d and $\wedge$ defined on each simplex are compatible on faces, and give $A^*(K)$ the structure of a dga (over $\mathbb{Q}$). We remark that the dga $A^*(K)$ is a remarkable compromise between flexibility and rigidity. Each $A^i(\Delta^n)$ is filtered by the *finite dimensional* vector spaces consisting of polynomial forms of bounded degree. On each simplex, these forms are rigid and one cannot appeal to partitions of unity. However, this rigidity does not propagate outside the simplex of domain of a form, and one can piece together and extend forms with great flexibility. For instance, any $\varphi \in A^*(\partial\Delta)$ can be extended to a form on $A^*(\Delta)$; this is a nontrivial (albeit elementary) algebraic fact. Consequently, any $\varphi \in A^*(K^n)$ extends to $\widetilde{\varphi}$ on $A^*(K)$.

Theorem 3.5 (PL de Rham theorem)**.** *The map $\rho : A^*(K) \to C^*(K;\mathbb{Q})$ defined by $\rho(\omega)(\sigma) = \int_\sigma \omega$ induces an algebra isomorphism on cohomology.*

Proof. We give only a hint of a proof. For details, see [2], Ch. 8.

First it is shown that $A^* \to C^*$ is surjective. To see this, observe that $\frac{1}{\mathrm{vol}(\Delta)} d\mathrm{vol}$ is in $A^*(\Delta)$ and restricts to 0 in $A^*(\partial\Delta)$. Thus we can extend it by zero to a form on $A^*(K^n)$ and in an arbitrary way to the higher dimensional skeleta.

Now, if $\varphi \in A^n$ is in the kernel of ρ, so is $d\varphi$; for, if σ is an $n+1$-simplex, then $\rho(d\varphi)(\sigma) = \int_\sigma d\varphi = \int_{\partial\sigma} \varphi = 0$. We therefore obtain a short exact sequence of (co)chain complexes $B^* \to A^* \to C^*$.

We claim that B^* is acyclic; equivalently, if $\varphi \in A^n$ satisfies $d\varphi = 0$ and $\rho(\varphi) = 0$ then there is $\psi \in A^{n-1}$ with $d\psi = \varphi$ and $\rho(\psi) = 0$. This is proved by induction; in the case $n = 1$, we can take $\varphi = q(t)dt$ on Δ^1. Then $\rho(\varphi) = 0$ means $\int_0^1 q dt = 0$. In this case, we can define $\psi(t) = \int_0^t q dt$.

For the general case, we remark that the proof would be easy if we were allowed to subdivide: each simplex Δ could be thought of as the cone from its boundary to the center. For a cone, a primitive for φ can be found by the Poincaré lemma, by defining ψ at a point p to be the integral of φ along a ray from the center to p. Instead we must think of each simplex as a cone from each top dimensional face to the opposite vertex, perform the integration trick for each face individually, then adjust the result inductively on the boundary to give the desired result.

Since B^* is acyclic, the short exact sequence $B^* \to A^* \to C^*$ gives rise to a long exact sequence in cohomology, proving that ρ induces an isomorphism on cohomology *groups*. Verifying that it is an isomorphism of rings amounts to taking a simplicial subdivision $(K \times K)'$ of $K \times K$, analyzing the map $A^*((K \times K)') \to C^*((K \times K)')$, and observing that the form $\varphi_0 \otimes \varphi_1$ on a product of simplices $\sigma_0 \times \sigma_1$ restricts to a polynomial form with rational coefficients on the subdivision. Then one compares with the usual definition of cup product as pull back of cross product under some suitable simplicial approximation of the diagonal $\Delta : K \to K \times K$. $\square$

Note that ρ is *not* a morphism of dgas, since C^* is not a dga.

3.3. **Minimal models.**

Definition 3.6. A dga A^* is *connected and simply-connected* if $H^0(A^*)$ is equal to the ground field, and $H^1(A^*) = 0$. A (connected and simply-connected) dga A^* is *minimal* if

(1) A^* is free as a graded-commutative algebra;
(2) $A^1 = 0$; and
(3) $d(A^*) \subset A^+ \wedge A^+$ where $A^+ = \oplus_{p>0} A^p$.

Condition (3) says that d is *decomposable*. In particular, generators of A^* are not in the image of d. Note further that the image of d is contained in $A^{\geq 4}$.

Definition 3.7. If A^* is a dga, a *minimal model* for A^* is a minimal dga $\mathcal{M}(A^*)$ together with a morphism of dgas $\rho : \mathcal{M}(A^*) \to A^*$ inducing an isomorphism on cohomology.

Definition 3.8. Let A^* be a dga. A *Hirsch extension* of A is an extension

$$A^* \to A^* \otimes_d \Lambda(V)_k$$

where V is a finite dimensional vector space of homogeneous degree k, $\Lambda(V)_k$ is the free graded-commutative algebra with unit generated by V, and $d : V \to A^{k+1} \cap \ker d$ is some differential. We extend d to the algebra $A^* \otimes_d \Lambda(V)_k$ by the Leibniz rule.

Providing $d(V)$ is in the kernel of d, the result is a dga (all that needs to be checked is that $d^2 = 0$). Note that $d : V \to A^{k+1}$ induces $d : V \to H^{k+1}(A^*)$; hence we can think of d as determining a class $[d] \in H^{k+1}(A^*; V^*)$.

In the sequel, we suppress the $*$ in the superscript of A, and the k in the subscript of V where this is not ambiguous.

Two Hirsch extensions $A \to A \otimes_d \Lambda(V)_k$ and $A \to A \otimes_d \Lambda(V')_k$ are equivalent if there is an isomorphism $\varphi : A \otimes_d \Lambda(V) \to A \otimes_d \Lambda(V')$ extending the identity on A.

Lemma 3.9. *Two Hirsch extensions $A \to A \otimes_d \Lambda(V)$ and $A \to A \otimes_d \Lambda(V')$ are equivalent iff there is an isomorphism $\psi : V \to V'$ so that $[dv] = [d'\psi(v)]$ in $H^{k+1}(A)$.*

Proof. For $v \in V$, write $\varphi(v) = a_v + \psi(v)$. Since $\varphi(a) = a$, ψ must be an isomorphism. Since $\varphi(dv) = d'(\varphi(v)) = d'a_v + d'\psi(v)$, we have $\varphi[dv] = [d'\psi(v)]$ in $H^{k+1}(A)$. Conversely, given an isomorphism $\psi : V \to V'$ with this property, $dv - d'\psi(v) = da_v$ for some a_v. Choose a_v in this way for some basis for V, define $\varphi(v) = a_v + \psi(v)$ and extend by linearity. □

Note that if A is free as a graded-commutative algebra, then so is any Hirsch extension. Now, suppose we build a dga by iterated Hirsch extensions, starting with some $\mathcal{M} = \mathcal{M}(0)$ equal to the base field, and where successive extensions are in degree $2, 3, 4$ and so on. Set $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes_d \Lambda(V)_{n+1}$. Since every dv has degree $n+2$, and since $\mathcal{M}(n)$ is (inductively) generated by elements of degrees $\leq n$, it follows that d is decomposable, and therefore $\mathcal{M} = \cup \mathcal{M}(n)$ is a minimal dga.

Conversely, every minimal dga arises this way:

Proposition 3.10. *Let $\mathcal{M}$ be minimal, and let $\mathcal{M}(n)$ be the subalgebra generated by elements of degrees $\leq n$. Then $\mathcal{M}(0) = \mathcal{M}(1) \subset \mathcal{M}(2) \subset \cdots$ with $\cup \mathcal{M}(n) = \mathcal{M}$, and with each $\mathcal{M}(n) \subset \mathcal{M}(n+1)$ a Hirsch extension.*

Proof. Since each $\mathcal{M}(i)$ is free as a graded-commutative algebra, $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes \Lambda(V)_{n+1}$ for some $\Lambda(V)$ in degree $n+1$, as graded-commutative algebras. Since $\mathcal{M}(1) = 0$ and d is decomposable, every dv is a sum of products of elements of degree $\leq n$; i.e. $dv \in \mathcal{M}(n)$. Hence each extension is a Hirsch extension. $\square$

Theorem 3.11. *Let A be a connected and simply-connected dga. Then A has a minimal model.*

Proof. We construct a minimal model $\mathcal{M}$ inductively as an increasing sequence of Hirsch extensions $\mathcal{M}(n) \to \mathcal{M}(n+1)$ together with maps $\rho_n : \mathcal{M}(n) \to A$ so that ρ_n^* is an isomorphism in degrees $\leq n$, and an injection in degree $n+1$. First, $\mathcal{M}(0)$ is the ground field; since A is connected and simply-connected, ρ_0 is an isomorphism in dimensions ≤ 1 and injective in dimension 2, so we may take $\mathcal{M}(1) = \mathcal{M}(0)$.

A map of chain complexes $f : C^* \to D^*$ induces a short exact sequence $D^* \to C^{*+1} \oplus D^* \to C^*$ where the middle terms become a chain complex with differential $d + f$ on C^* and $-d$ on D^*. There is an associated long exact sequence in cohomology, whose middle terms are the *relative cohomology* of f. Note $H^n(D) \to H^{n+1}(C, D) \to H^{n+1}(C)$ because of the degree shift.

Assume we have constructed $\rho_n : \mathcal{M}(n) \to A$. Hence the relative cohomology $H^i(\mathcal{M}(n), A)$ vanishes for $i \leq n+1$. Let $V = H^{n+2}(\mathcal{M}(n), A)$. We will define a Hirsch extension $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes_d \Lambda(V)_{n+1}$. Each $v \in V$ has a representative cocycle $(m_v, a_v) \in \mathcal{M}(n)^{n+2} \oplus A^{n+1}$. The cocycle property means $dm_v = 0$ and $\rho_n(m_v) = da_v$. So we define the extension by setting $dv = m_v$ on a basis, and then define ρ_{n+1} by setting it equal to ρ_n on $\mathcal{M}(n)$ and $\rho_{n+1}(v) = a_v$. Since $d^2(v) = d(m_v) = 0$, $\mathcal{M}(n+1)$ is a Hirsch extension. Since $\rho_n(dv) = \rho_n(m_v) = da_v = d(\rho_{n+1}(v))$, this is a morphism of dgas.

Now, by construction, $H^{n+2}(\mathcal{M}(n), \mathcal{M}(n+1)) = V$ with representative cocycles of the form (dv, v), and $H^{n+3}(\mathcal{M}(n), \mathcal{M}(n+1)) = 0$ because $\mathcal{M}(n)$ has no elements of degree 1, and therefore $\mathcal{M}(n)$ and $\mathcal{M}(n+1)$ are the same in degree $n+2$.

The map of pairs $(\mathcal{M}(n), \mathcal{M}(n+1)) \to (\mathcal{M}(n), A)$ gives rise to a map of cohomology long exact sequences. The five lemma now proves the induction step. The theorem follows by taking direct limits. $\square$

3.4. **Homotopy theory of dgas.** Let (t, dt) denote the free dga over $\mathbb{Q}$ generated by t in degree 0 and dt in degree 1; hence (t, dt) consists of $\mathbb{Q}$ polynomial forms on $\mathbb{R}$. We denote by $(t, dt)|0$ and $(t, dt)|1$ the restrictions to the dga of "polynomial forms" at the points 0, 1 in $\mathbb{R}$ respectively (of course, these algebras are just $\mathbb{Q}$, and the restriction maps are given by evaluating t to 0 or 1 and sending dt to 0). For a dga B and $\beta \in B \otimes (t, dt)$ denote the image of β in $B \otimes (t, dt)|0 = B$ by $\beta|0$, and similarly with 1 in place of 0.

Definition 3.12. Let f and g be two maps between dgas A and B. A *homotopy* from f to g is a map of dgas

$$H : A \to B \otimes (t, dt)$$

satisfying $H|0 = f$ and $H|1 = g$. That is, $A \to B \otimes (t, dt) \to B \otimes (t, dt)|0 = B$ agrees with f, and $A \to B \otimes (t, dt) \to B \otimes (d, dt)|1 = B$ agrees with g.

Define maps

$$\int_0^1 : B \otimes (t, dt) \to B$$

by $\int_0^1 b\otimes t^i = 0$ and $\int_0^1 b\otimes t^i dt = (-1)^{|b|}b/(i+1)$. Similarly, define

$$\int_0^t : B\otimes(t,dt)\to B\otimes(t,dt)$$

by $\int_0^t b\otimes t^i = 0$ and $\int_0^t b\otimes t^i dt = (-1)^{|b|}b\otimes t^{i+1}/(i+1)$.

Lemma 3.13. *For any $\beta\in B\otimes(t,dt)$ we have*

$$d\left(\int_0^t \beta\right) + \int_0^t d\beta = \beta - (\beta|0)\otimes 1$$

Moreover, if $H : A\to B\otimes(t,dt)$ is a homotopy from f to g then

$$d\int_0^1 H(a) + \int_0^1 dH(a) = g(a) - f(a)$$

Proof. Note that the second formula follows from the first by taking $\beta = H(a)$ and restricting to 1. To prove the first formula,

$$d\left(\int_0^t b\otimes t^i\right) + \int_0^t (db\otimes t^i + (-1)^{|b|}b\otimes it^{i-1}dt) = \begin{cases} b\otimes t^i & \text{if } i>0 \\ 0 & \text{else}\end{cases}$$

and

$$d\left(\int_0^t b\otimes t^i dt\right) + \int_0^t db\otimes t^i dt = (-1)^{2|b|}b\otimes t^i dt = b\otimes t^i dt$$

(the two terms of the form $\pm db\otimes t^{i+1}/(i+1)$ cancel). $\square$

Proposition 3.14. *Given a diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow & & \downarrow \varphi \\ A\otimes_d \Lambda(V)_k & \xrightarrow{f} & C \end{array}$$

and a homotopy $H : A\to C\otimes(t,dt)$ from φg to $f|A$, the obstruction to extending g to $\widetilde{g} : A\otimes_d\Lambda(V)_k\to B$ and extending H to $\widetilde{H}$ from f to $\varphi\widetilde{g}$ is represented by a class $[o]\in H^{k+1}(B,C;V^*)$.

Proof. Given $v\in V$, define $o(v)\in B^{k+1}\oplus C^k$ by $o(v) = (g(dv), f(v) + \int_0^1 H(dv))$. First of all, this is a cocycle, since

$$do(v) = (dg(dv), \varphi g(dv) - df(v) - d\int_0^1 H(dv))$$

Since H is a morphism of dgas, $dH(dv) = 0$ so this is equal to

$$= (g(d^2v), \varphi g(dv) - f(dv) - d\int_0^1 H(dv) - \int_0^1 dH(dv)) = (0,0)$$

where we use the defining fact that H is a homotopy from $f|A$ to φg on A, and Lemma 3.13.

We now think of the class of o as a homomorphism from V to $H^{k+1}(B,C)$. If $[o(v)] = 0$ for all v, then there are cochains (b_v, c_v) so that $d(b_v,c_v) = o(v)$.

This means first of all that $db_v = g(dv)$, and second of all that $\varphi(b_v) - dc_v = f(v) + \int_0^1 H(dv)$, or equivalently that

$$dc_v = \varphi(b_v) - f(v) - \int_0^1 H(dv)$$

We define $\widetilde{g}(v) = b_v$, and

$$\widetilde{H}(v) = f(v) + \int_0^t H(dv) + d(c_v \otimes t)$$

on a basis v, and extend to V. We claim these extend to maps of dgas $\widetilde{g} : A \otimes_d \Lambda(V)_k \to B$ extending g and $\widetilde{H} : A \otimes_d \Lambda(V)_k \to C \otimes (t, dt)$ extending H. Since the extension exists as maps of algebras, we just need to check that it respects the differentials; i.e. that $d\widetilde{g}(v) = g(dv)$ and $d\widetilde{H}(v) = H(dv)$ on a basis element v. Now, $d\widetilde{g}(v) = db_v = g(dv)$ by the definition of $\widetilde{g}$ and b_v. Moreover,

$$d\widetilde{H}(v) = df(v) + d\int_0^t H(dv) = df(v) + H(dv) - H(dv)|0$$

where the second equality follows from $dH(dv) = H(d^2v) = 0$ and Lemma 3.13. But $H(dv)|0 = f(dv)$ so $d\widetilde{H}(v) = H(dv)$ as claimed.

Finally, we need to check that $\widetilde{H}$ is a homotopy from f to $\varphi\widetilde{g}$. This is just a calculation; first, $\widetilde{H}(v)|0 = f(v)$, and second,

$$\begin{aligned}\widetilde{H}(v)|1 &= f(v) + \int_0^1 H(dv) + dc_v \\ &= f(v) + \int_0^1 H(dv) + \left(\varphi(b_v) - f(v) - \int_0^1 H(dv)\right) \\ &= \varphi(b_v) = \varphi\widetilde{g}(v)\end{aligned}$$

Conversely, if $\widetilde{g}$ and $\widetilde{H}$ are any extensions as above, we can define $\psi : V \to B^k \oplus C^{k-1}$ by $\psi(v) = (\widetilde{g}(v), \int_0^1 \widetilde{H}(v))$, which satisfies $d\psi = (g(dv), f(v) + \int_0^1 H(dv)) = o(v)$. □

We would like to apply this to a diagram like

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\varphi} & A & \xrightarrow{\mu} & C \\ \downarrow & & \downarrow f & & \| \\ \mathcal{M} \otimes_d \Lambda(V)_n & \xrightarrow{\psi} & B & \xrightarrow{\nu} & C \end{array}$$

Satisfying properties

(1) μ is onto;
(2) $\mu\varphi = \nu\psi|\mathcal{M}$;
(3) the composition of the homotopy $H : \mathcal{M} \to B \otimes (t, dt)$ with $\nu \otimes 1 : B \otimes (t, dt) \to C \otimes (t, dt)$ is constant; i.e. the composition maps $m \in \mathcal{M}$ to $c_m \otimes 1$.

Lemma 3.15. *In the situation above, the obstruction class $o \in H^{n+1}(A, B; V^*)$ vanishes if and only if there is an extension $\widetilde{\varphi}$ of φ and an extension $\widetilde{H} : \mathcal{M} \otimes_d \Lambda(V)_n \to B \otimes (t, dt)$ for which $\mu\widetilde{\varphi} = \nu\psi$ and $(\nu \otimes 1)\widetilde{H}$ is constant.*

This is proved by examining the extensions constructed in Proposition 3.14. Applying this lemma with $C = B$, we obtain the following corollary:

Corollary 3.16. *Given a commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & A \\ \Big\downarrow & & \Big\downarrow f \\ \mathcal{M} \otimes_d \Lambda(V)_n & \xrightarrow{\psi} & B \end{array}$$

for which f is onto, the class $o : V \to H^{n+1}(A, B)$ is the obstruction to extending φ to $\widetilde{\varphi} : \mathcal{M} \otimes_d \Lambda(V)_n \to A$ so that $f\widetilde{\varphi} = \psi$.

Corollary 3.17. *If $\mathcal{M}$ is minimal, the relation on maps $\mathcal{M} \to A$ of being homotopic is an equivalence relation.*

Proof. Suppose $H : \mathcal{M} \to A \otimes (t_1, dt_1)$ is a homotopy from f_0 to f_1, and $J : \mathcal{M} \to A \otimes (t_2, dt_2)$ is a homotopy from f_1 to f_2. Let C be the dga with generators t_1, t_2 in degree 0, dt_1, dt_2 in degree 1, and relations $t_2(t_1 - 1) = 0$, $t_1 dt_2 = t_2 dt_1 = 0$. Geometrically, C consists of rational forms on the variety $t_2(t_1 - 1) = 0$ in the t_1-t_2 plane.

From H and J we can build a map $H + J : \mathcal{M} \to A \otimes C$. We would like to lift this to $A \otimes (t_1, t_2, dt_1, dt_2)$ and thence to $A \otimes (t, dt)$ by the map that restricts forms on the t_1, t_2 plane to the line $t_1 = t_2$:

$$(t_1, t_2, dt_1, dt_2) \to (t_1 = t_2, dt_1 = dt_2) = (t, dt)$$

We lift the map inductively on $\mathcal{M}(n)$; it suffices to show that given a lift on $\mathcal{M}(n)$, we can build a lift on a Hirsch extension $\mathcal{M}(n + 1) = \mathcal{M}(n) \otimes \Lambda(V)_{n+1}$. Since $A \otimes (t_1, t_2, dt_1, dt_2) \to A \otimes C$ is surjective, Corollary 3.16 says that obstruction to the lift lies in $H^*(A \otimes (t_1, t_2, dt_1, dt_2), A \otimes C)$. But inclusion of the variety $t_2(t_1 - 1) = 0$ into the t_1-t_2 plane is a homotopy equivalence, and therefore the relative cohomology vanishes, so the lift exists. □

Given a dga A and a minimal dga $\mathcal{M}$, let $[\mathcal{M}, A]$ denote the set of homotopy classes of maps $\mathcal{M} \to A$.

Theorem 3.18. *Let $\varphi : B \to C$ induce an isomorphism on cohomology, and let $\mathcal{M}$ be a minimal dga. Then $\varphi_* : [\mathcal{M}, B] \to [\mathcal{M}, C]$ is a bijection.*

Proof. Given $f : \mathcal{M} \to C$ we lift it inductively on $\mathcal{M}(n)$ to B. At every step the obstruction lies in $H^{n+2}(B, C; V^*)$ where $\mathcal{M}(n+1) = \mathcal{M} \otimes \Lambda(V)_{n+1}$. Since $B \to C$ induces an isomorphism in cohomology, these obstructions all vanish, and the lift can be constructed. This shows φ_* is surjective.

To show it is injective, suppose we have $f_0, f_1 : \mathcal{M} \to B$ and a homotopy $H : \mathcal{M} \to C \otimes (t, dt)$ between φf_0 and φf_1. We lift H to $\mathcal{M} \to B \otimes (t, dt)$ extending the lifts f_i of φf_i. □

Corollary 3.19. *Let $\rho_A : \mathcal{M}_A \to A$ and $\rho_B : \mathcal{M}_B \to B$ be minimal models, and let $f : A \to B$ be a map of dgas. There is a map $\hat{f} : \mathcal{M}_A \to \mathcal{M}_B$ and a homotopy from $\rho_B \hat{f}$ to $\hat{f} \rho_A$. Moreover, $\hat{f}$ is determined up to homotopy by these properties.*

Proof. Since ρ_B induces an isomorphism on cohomology, the map $f \rho_A : \mathcal{M}_A \to B$ lifts to $\hat{f} : \mathcal{M}_A \to \mathcal{M}_B$ by Theorem 3.18. □

Lemma 3.20. *Let $I : \mathcal{M} \to \mathcal{M}'$ be a map of minimal dgas inducing an isomorphism on cohomology. Then I is an isomorphism.*

Proof. Evidently I induces $I_n : \mathcal{M}(n) \to \mathcal{M}'(n)$. We prove the result by induction on n. The inclusions $\mathcal{M}(n) \to \mathcal{M}$ and $\mathcal{M}'(n) \to \mathcal{M}'$ give rise to long exact sequences in cohomology which are related by I. By the five lemma and induction on n, we conclude that $H^{n+2}(\mathcal{M}(n), \mathcal{M}) \to H^{n+2}(\mathcal{M}'(n), \mathcal{M}')$ is an isomorphism.

On the other hand, $H^{n+2}(\mathcal{M}(n), \mathcal{M}) = H^{n+2}(\mathcal{M}(n), \mathcal{M}(n+1)) = V_{n+1}$ so $I : V_{n+1} \to V'_{n+1}$ is an isomorphism, and by the structure theorem for Hirsch extensions the induction step is proved. □

Finally we obtain the main theorem of this section:

Theorem 3.21 (Uniqueness of minimal models)**.** *If A is a dga, and $\rho : \mathcal{M} \to A$, $\rho' : \mathcal{M}' \to A$ are minimal models for A, there is an isomorphism $I : \mathcal{M} \to \mathcal{M}'$ and a homotopy H from ρ to $\rho' I$. Moreover I as above is unique up to homotopy.*

Proof. Since $\rho : \mathcal{M}' \to A$ induces isomorphisms in cohomology, $[\mathcal{M}, A] = [\mathcal{M}, \mathcal{M}']$ by Theorem 3.18, so there is some morphism $I : \mathcal{M} \to \mathcal{M}'$ for which $\rho' I = \rho$, such that I is unique up to homotopy.

By construction, I induces an isomorphism on cohomology. So after applying Lemma 3.20 we are done. □

3.5. **Homotopy theory of dgas and rational homotopy theory.** Suppose B is a simplicial complex, and $p : E \to B$ is a principal fibration with fiber $K(\pi, n)$. Then in the Serre spectral sequence with coefficients in π, we have $E_2^{p,q} = 0$ for $0 < q < n$ and $E_2^{0,n} = H^n(K(\pi, n); \pi) = \mathrm{Hom}(\pi, \pi)$. Hence $E_2^{0,n} = E_r^{0,n}$ for $r \le n$ and the first nonzero differential is the transgression $d_{n+1} : E_n^{0,n} \to E_n^{n+1,0} = H^{n+1}(B; \pi)$, and if id denotes the identity element of $\mathrm{Hom}(\pi, \pi)$, then $d_{n+1}(\mathrm{id}) \in H^{n+1}(B; \pi)$ is the k-invariant.

Taking $\mathbb{Q}$ coefficients instead we get

$$d_{n+1} : \mathrm{Hom}(\pi \otimes \mathbb{Q}, \mathbb{Q}) = \mathrm{Hom}(\pi, \mathbb{Q}) \to H^{n+1}(B; \mathbb{Q}) = \mathrm{Hom}(H_{n+1}(B), \mathbb{Q})$$

which is dual to some class $[d_{n+1}] \in H^{n+1}(B; \pi \otimes \mathbb{Q}) = \mathrm{Hom}(H_{n+1}(B), \pi \otimes \mathbb{Q})$. If π is already a finite dimensional $\mathbb{Q}$ vector space, then $[d_{n+1}] \in H^{n+1}(B; \pi)$ is the k-invariant of the fibration.

On the algebraic side, given a finite dimensional $\mathbb{Q}$ vector space π and an element $[d] \in H^{n+1}(B; \pi)$ there is a Hirsch extension $A^*(B) \otimes_d \Lambda(\pi^*)_n$ unique up to isomorphism, where $A^*(B)$ denotes the (rational) PL de Rham complex of B. If E were a simplicial complex, we would expect to relate this dga to $A^*(E)$, and obtain a map $\rho : A^*(B) \otimes_d \Lambda(\pi^*)_n \to A^*(E)$ inducing an isomorphism in cohomology.

Of course E is not a simplicial complex; the method of *simplicial models* is a technical workaround to approximate E by a simplicial complex. In any case, one obtains in this way two natural transformations. Let $\mathfrak{P}(B)$ denote the set of equivalence classes of principal fibrations over B with fiber a $K(\pi, n)$ (for various n) with $\mathrm{Hom}(\pi, \mathbb{Q})$ finite dimensional, and let $\mathfrak{P}_{\mathbb{Q}}(B)$ be the subset for which π is a $\mathbb{Q}$ vector space. Let $\mathfrak{H}(B)$ denote the set of equivalence classes of finite dimensional Hirsch extensions of $A^*(B)$. Then there are natural transformations $\mathfrak{P}(B) \to \mathfrak{H}(B)$ and $\mathfrak{H}(B) \to \mathfrak{P}_{\mathbb{Q}}(B)$ for which the composition $\mathfrak{H}(B) \to \mathfrak{H}(B)$ is the identity, and $\mathfrak{P}(B) \to \mathfrak{P}_{\mathbb{Q}}(B)$ is localization.

If A is a dga, by abuse of notation we let $\mathfrak{H}(A)$ denote the set of equivalence classes of finite dimensional Hirsch extensions of A.

Lemma 3.22. *Let A and B be dgas, and suppose $f : A \to B$ induces an isomorphism on cohomology. Then f induces a bijection $\mathfrak{H}(A) \to \mathfrak{H}(B)$.*

Proof. This is immediate from Lemma 3.9. □

Corollary 3.23. *Let $\mathcal{M} \to A^*(B)$ be a minimal model. Then there is a natural bijection $\mathfrak{H}(\mathcal{M}) \to \mathfrak{P}_{\mathbb{Q}}(B)$.*

Theorem 3.24 (Minimal models recover rational homotopy type)**.** *Let X be a simply connected simplicial model. Let $\cdots X_3 \to X_2$ be a Postnikov tower for X, and let $\cdots X_3' \to X_2'$ be a simplicial model for the Postnikov tower. Finally, let $\mathcal{M} = \cup_n \mathcal{M}(n)$ be the minimal model for X obtained by a sequence of Hirsch extensions associated to the tower of principal fibrations. Then*

1. *$\mathcal{M}(n)$ is the minimal model for X_n', and therefore $H^*(\mathcal{M}(n)) = H^*(X_n; \mathbb{Q})$; and*
2. *if $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes_d V_{n+1}$ then $V_{n+1} = \mathrm{Hom}(\pi_{n+1}; \mathbb{Q})$, and the rational k-invariant $k_{n+2} \in H^{n+2}(X_n; \pi_{n+1} \otimes \mathbb{Q})$ is equal to $[d] \in H^{n+2}(\mathcal{M}(n); V^*)$.*

Proof. We give the sketch of a proof. For details, see [2], pp. 135–138.

The construction of the minimal model $\mathcal{M}$ inductively from the data of the Postnikov tower is straightforward. There is some work in obtaining a map $\rho : \mathcal{M} \to A^*(X)$ inducing an isomorphism on cohomology; each map $f_n : X \to X_n'$ can be made simplicial after subdividing X. Subdivision induces compatible maps between PL de Rham complexes, and one obtains a dga $A^*_{PL}(X)$ as a direct limit under subdivision. Each $\rho_n : \mathcal{M}(n) \to A^*(X_n')$ defines a map of $\mathcal{M}(n)$ to $A^*_{PL}(X)$. These maps are not *a priori* compatible under restriction, but can be adjusted by homotopies, using Corollary 3.16. Inductively, we exhibit $\mathcal{M}$ as a minimal model for $A^*_{PL}(X)$, and since $A^*(X) \to A^*_{PL}(X)$ induces an isomorphism in cohomology, by Theorem 3.18 there is a map $\mathcal{M} \to A^*(X)$ exhibiting $\mathcal{M}$ as a minimal model for $A^*(X)$. □

In principle, for a finite simplicial complex X one can construct the PL de Rham complex $A^*(X)$ explicitly, or at least enough of it to capture the cohomology of X. Then the minimal model can also be constructed explicitly by the method in Theorem 3.11, and we thereby obtain an algorithm to compute the rational homotopy type of X.

Example 3.25 (Formality)*.* A dga is *formal* if it has the same minimal model as a dga with vanishing d; i.e. if the dga has the same minimal model as its cohomology.

A space is formal if its PL de Rham complex is formal. For such spaces, the rational homotopy type can be determined easily. Examples are spheres, H-spaces, symmetric spaces (e.g. Grassmannians and projective spaces), and compact Kähler manifolds (which include all nonsingular projective varieties). Formality is preserved under taking wedges and direct products, and (for manifolds) under connect sums. See e.g. [1] for more details.

References

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