ALGEBRAIC TOPOLOGY, FALL 2016, FINAL

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This final exam was posted online on Friday, December 2, and is due by 11:30 on Friday, December 9. Collaboration is not allowed, nor is the use of outside materials and textbooks. Hatcher and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Bonus problems are just for fun, and do not contribute to the grade for the class.

Problem 1. Compute the cup product structure on the cohomology of a Klein bottle, for both integer and $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Problem 2. Give an example of a connected CW complex X with $\pi_1(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i > 0$ but such that X is not homotopy equivalent to a circle. (Hint: take the 2-skeleton $X^2 = S^2 \vee S^1$ and think about how to kill H_2 without killing π_2)

Problem 3. Let M be a closed (i.e. compact without boundary), connected, simply-connected 3-manifold. Show that M is homotopy equivalent to S^3. (Bonus question: show M is homeomorphic to S^3)

Problem 4.
(1) What is the cup product structure on the cohomology of $S^m \times S^n$?
(2) Use the previous result to show that $\pi_{m+n-1}(S^m \vee S^n)$ is nonzero.

Problem 5. Let M be a closed, connected, nonorientable 3-manifold. Show that there is a nontrivial homomorphism $\pi_1(M) \to \mathbb{Z}$. (hint: what is the Euler characteristic of M?)

Problem 6. Is there a map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ of negative degree (i.e. taking the generator of $H_4(\mathbb{CP}^2)$ to a negative multiple of itself)?

Problem 7. A closed orientable n-manifold M is *spherical* if there is some map $f : S^n \to M$ of nonzero degree (i.e. for which the image of the generator of $H_n(S^n)$ is equal to a nonzero multiple of the generator of $H_n(M)$). Prove that if M is spherical and $n > 1$, then $\pi_1(M)$ is finite.

Problem 8.
(1) Consider the fibration $SO(3, \mathbb{R}) \to S^2$ coming from the standard action by rotations on the unit sphere in $\mathbb{R}^3$. What is the map $\pi_2(S^2) \to \pi_1(S^1)$ in the long exact sequence of homotopy groups?
(2) Let M be a closed 3-manifold for which there is a fibration $M \to S^2$ with S^1 as the fiber. What can you say about $\pi_1(M)$? Give at least 4 different examples of such M.

Problem 9. Let M be a closed, simply-connected 4-manifold. Let W be obtained from M by removing a point. Show that W is homotopy equivalent to a wedge of S^2s. (Bonus question: show that M can be obtained from a wedge of 2-spheres up to homotopy by attaching a 4-cell, and describe how the element of $\pi_3(\vee_i S_i^2)$ given by the attaching map determines the cup product pairing on $H^2(M; \mathbb{Z})$)

Bonus Problem 10. Show that $\mathbb{CP}^2$ is not the boundary of a (smooth) compact 5-manifold.

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