Building surfaces

Danny Calegari

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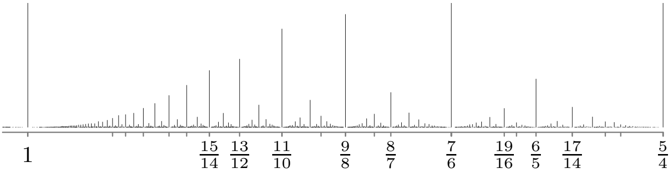
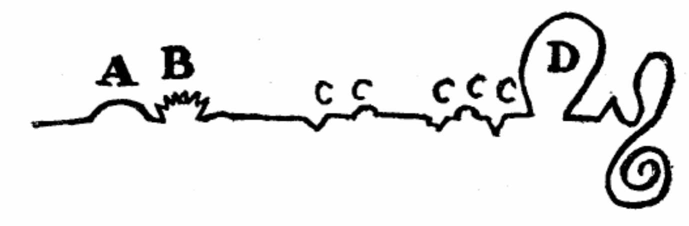


1. Surfaces

Let X be a space. Classically, X is studied by looking at maps from 1-manifolds to X.



Lawrence Sterne, *Tristram Shandy*

In the Riemannian world, we get geodesics, Jacobi fields, Morse theory, etc.

In the topological world, we get the fundamental group(oid), covering spaces, etc.

It is a relatively recent idea to study X by looking at maps from surfaces to X (this is morally a kind of "complexification" of geometry).

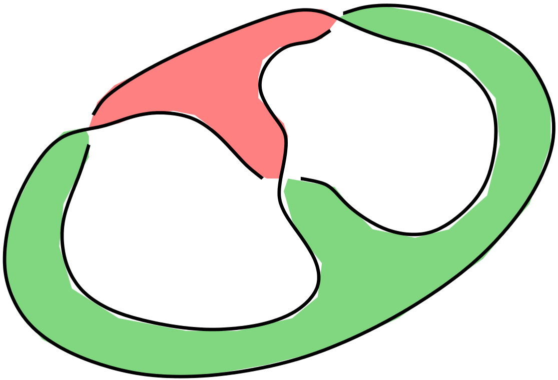
after Frank Morgan and Sir John Tenniel

In the Riemannian world, we get minimal surfaces, (pseudo-) holomorphic curves, string theory, etc.

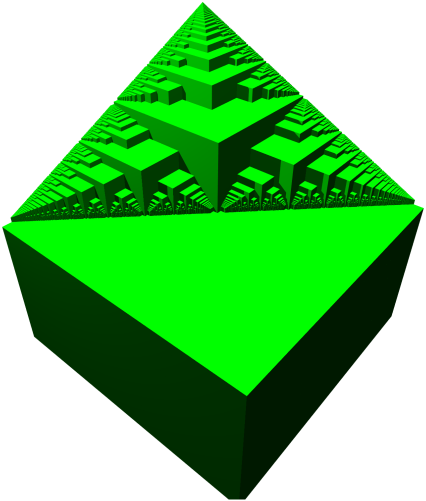
I would like to discuss the topological side of this story.

I am interested in questions like:

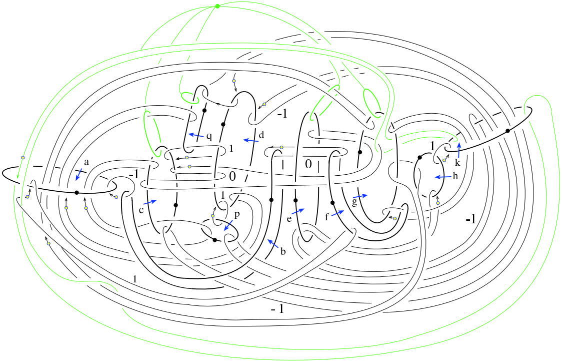
Question: What is the simplest spanning surface of a knot?



Question: Which Seifert fibered spaces admit taut foliations?



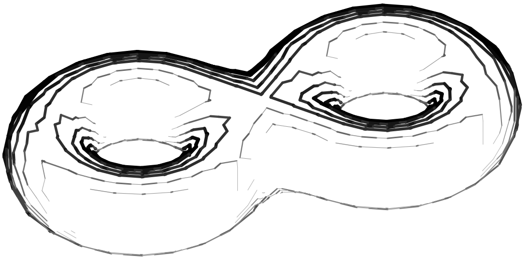
Question: How small can $\frac{\text{rank } H_2}{\text{signature}}$ be for a smooth, simply connected 4-manifold?



S. Akbulut, *The Akhmedov-Park exotic* $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$

Fundamental Geometric Question: If L is a 1-manifold in X, what is the simplest surface $S \to X$ with $\partial S \to L$?

"Simplest" here is measured topologically by the genus.



a closed surface of genus 2

This is equivalent to a problem in group theory for the fundamental group $G = \pi_1(X)$.

If $x, y \in G$, the commutator of x and y, denoted $[x, y]$, is the element $xyXY \in G$.

Fundamental Algebraic Question: If G is a group, and $g \in [G, G]$, what is the least number of commutators in G whose product is g?

This is called the commutator length of g, and denoted $\mathrm{cl}(g)$.

This question is just too hard to answer in general.

Example (Ore Conjecture): If G is a finite (noncyclic) simple group, $\mathrm{cl}(g) = 1$ for $g \in G - \mathrm{id}$. (Recently proved by LOST).

Example: In a free group $F = \langle a, b \rangle$,

$$\mathrm{cl}([a,b]) = 1, \quad \mathrm{cl}([a,b]^2) = 2, \quad \mathrm{cl}([a,b]^3) = 2$$

This follows from Culler's identity

$$[a,b]^3 = [abA, BabAA][Bab, bb]$$

More complicated identities hold, like

$$baBABAbaBabA^3 = [baBa, AABAbaBabAbaBABAbaBababABBa][AABAbaa, ABabAbaBABAbbaa][ABabABabab, aBAAba]$$

Extra structure emerges when we stabilize.

Definition: The stable commutator length of g is the limit

$$\mathrm{scl}(g) := \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$$

Geometrically, if X is a space with $\pi_1(X) = G$ and g is represented by L, then

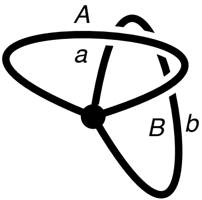
$$\mathrm{scl}(g) = \inf \frac{-\chi(S)}{2n}$$

over all $S \to X$ for which $\partial S \to L$ is an n-fold cover.

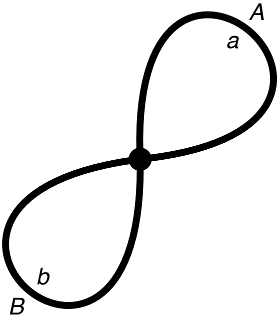
A surface realizing the infimum is extremal.

Remark: genus is not multiplicative under covers, but χ is.

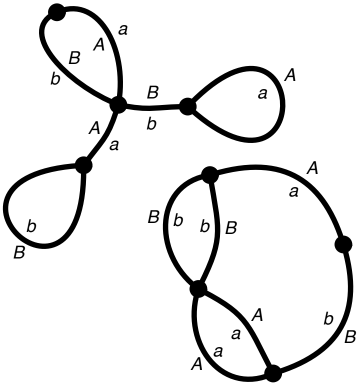
Example: An extremal surface for $abAB$ in $\langle a, b\rangle$ is a one-holed torus. So $-\chi = 1$ and $n = 1$, and $\mathrm{scl}(abAB) = 1/2$.



Example: An extremal surface for $a + b + AB$ in $\langle a, b\rangle$ is a pair of pants. So $-\chi = 1$ and $n = 1$, and $\mathrm{scl}(a + b + AB) = 1/2$.



Example: An extremal surface for $abAB + a + b + AB$ in $\langle a, b \rangle$ has two components, each a 4-holed sphere. So $-\chi = 4$ and $n = 3$, and $\mathrm{scl}(abAB + a + b + AB) = 2/3$.

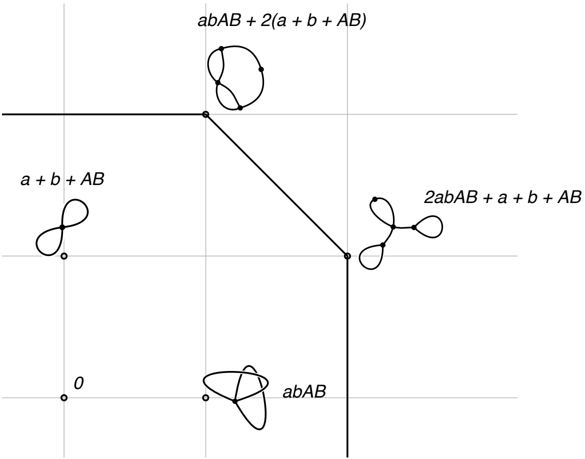


Definition: Let $B(G)$ be the vector space of finite linear combinations of elements in G representing 0 in $H_1(G)$. Define

$$B^H(G) = B(G)/\langle g^n - ng, g - hgh^{-1} \rangle$$

scl is a (possibly degenerate) norm on $B^H(G)$.

Example: Part of the unit ball in the subspace of $B^H(\langle a, b\rangle)$ spanned by $abAB$ and $a+b+AB$. Extremal surfaces associated to vertices of the unit ball can always be taken to be connected.



2. Free groups

Free groups are an important special case for the study of scl, for many reasons:

1. Free groups are essentially the *only* nontrivial example where exact calculations are possible!
2. $\pi_1(S)$ is free for S a surface with boundary; scl in free groups is a tool to study the category of surfaces and homotopy classes of maps between them;
3. Free groups are a model for more general classes of "negatively curved" groups.

The analysis begins with the Rationality Theorem:

Rationality Theorem: Let F be a free group.

1. The unit ball in the scl norm on $B^H(F)$ is a rational polyhedron;
2. Every rational $b \in B^H(F)$ admits an extremal surface;
3. There is a (fast!) polynomial time algorithm to compute scl.

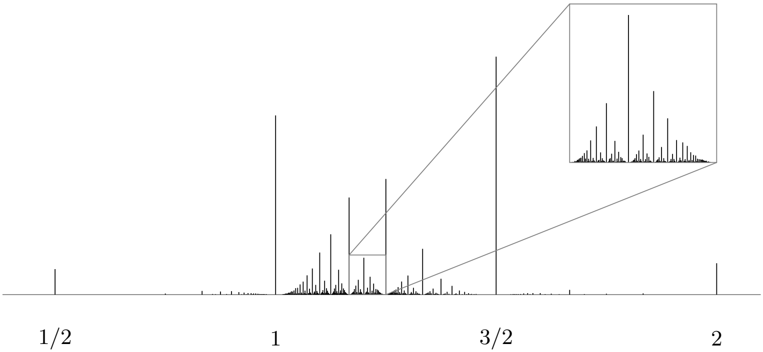
The algorithm is so fast, it can be practically implemented. This makes possible computer experiments, which uncover a range of unsuspected phenomena.

Program `scallop` available at `github.com/aldenwalker`.

scl of some random words of length 50 in rank 2:

scl (BaBBaabAAbABaBaabaaaaaBaaBaBAbAbbABAbAAAABAbbbbAAB) = 8841/4951 = 1.7857
scl (bbABabAABBaaBaaBabAAbbabbaaBBAAbAbbaBABABABABBaabb) = 2 = 2
scl (aBAbAAbbABBBaaBBaaaBABaBaabAbAAAbaBaBBAbaabbbAbAAb) = 13349/7314 = 1.82513
scl (bababAbaaBBabbaBABBABBabbbAAbAAABBABAbaabABaaBaBAb) = 32141/17414 = 1.8457
scl (baBAABabbbAAABBAABaBBaaaabbAABabaaBBabAbbbABBABabb) = 16759/8112 = 2.06595
scl (AbaBaaabbAABBBBBBAABBaaababaBBAAAAbbbbAbbABaBababA) = 6443/3340 = 1.92904
scl (AABBAABBBBAbAbaBBaBBBAbbbbbbbAABaBaBabbaBaBabbabbb) = 391/218 = 1.79358
scl (ABabbbAbbaabaaBAAbbbbaBBBBAABabaaBBBBabbbAbABBBAAB) = 9025/4918 = 1.8351
scl (bbbbABaBaBAABBABBaBAbbbAbaaaBBAbabAbAbABaBaaaBBAbb) = 709/343 = 2.06706
scl (aaBaabaBaabAbAbbAABAAABBABABBAbbaabAbbaaBaBAABBabb) = 153/91 = 1.68132
scl (BBabbbABabAbabaaBABaabAbabbAAbbaaaaBABABBaBAABABAA) = 58573/31764 = 1.84401
scl (AbAAAAAAbAbABBBababbbabaaBaaabaBAbaBaBAAAbaaBBABaB) = 21427/10786 = 1.98656
scl (BaaabaBaabAbabABAbAbaBBBBBaaabaaBAAAAbaaBAAAbbAAAA) = 29483/14764 = 1.99695
scl (bbaBBAAABAbAAAbabbabAbaBABAbaBBABaaaaaBABBABababab) = 167/95 = 1.75789
scl (BABabAbaBabbaaabaBABBBabaaBAbABAbbAbabbaBABAABBAbA) = 2003/1164 = 1.72079
scl (AbbAbAbaaBBababbaBaBBaBBAABabbAbabaBABBAbbbaBBAABA) = 87/46 = 1.8913
scl (aaBBBABBBabbaabAbAAbbbaaBaBABaBAAAbAbababbAbbABBBB) = 641/347 = 1.84726
scl (BabABBBBBaaabABBabAbAbbABabAbaaBAAbaBabAAbbbaBAABa) = 539/302 = 1.78477
scl (AAbbaBAABBBBBBabaBAAbbbbAbabABBABBaaaBabaBAbbaabAb) = 57/32 = 1.78125
scl (baaaBABaBAbbABBaaaaBAAbabbAbAABBaBBABAbAAbaabbbaBA) = 14401/7242 = 1.98854
scl (BaBabAbAAbaaBBBabaBBAAbAbAAAABABaaaaaaaBAAAbabbAba) = 3841/2018 = 1.90337
scl (AbAbAbAAbAbaBaaBBaaBBBABAbbaBaaBBABAbaabbABBAbaabb) = 239/115 = 2.07826
scl (bABBaaaabABBBABBBBaBAAbABaabABabAbbbbAbbAAbaBBabab) = 2879/1478 = 1.9479
scl (ABBAbbbaBABBaabaaabbbAbaBabAbAAAAbbbbaBABABBaBaBBB) = 497/242 = 2.05372
scl (abbABAAAbAbaaBBABAABBBaaBabbabAbaaaabaabaBABAAABAb) = 2909/1538 = 1.89142
scl (AABBABBaaBababAAAbbaaaBAABBaaabbbbAbbaaaBAABAAAbaB) = 706/343 = 2.05831
scl (abbAbbabbAAAbABaBabbbbbaaBaBAABBBBBAAAbaaBBBABabaB) = 6413/3460 = 1.85347
scl (ABAbAAbaBaaababbabAbbaBBABBaabABabAbAbaBBABabABABB) = 471/241 = 1.95436
scl (babaabAAABaababbAAAbABBABAAAbaBABBaaBABaaaBaabAAba) = 14601/8134 = 1.79506
scl (AABaabAbAABabAbbaBAABaBBBaBBAbaBaabABaaBBAbabbbbAb) = 175/94 = 1.8617
scl (aaababAbaBBBABBaBBAABBaaaBAbabbAbAAAAbabbbbAAABBaa) = 542258/286293 = 1.89407
scl (aBBBAbaBaaabABABBAAAABABAbbabbbbbbbaaBBaaBabAAABab) = 7589/3781 = 2.00714
scl (AbAABBaBBABBBabbabbbABBBAbbABBabbbbAbaBaabaBaBabAA) = 16697/8624 = 1.93611
scl (BaBAABBAbABBBaabaaaabbABABBAABBaabAbabbbbbaaBAbbAA) = 3011/1592 = 1.89133

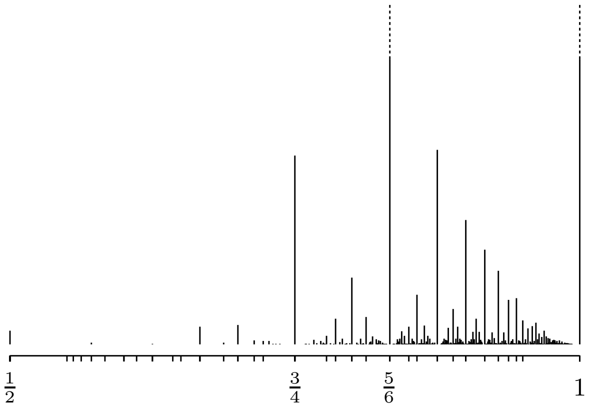
Histogram of scl on random words of length 32 in rank 2:



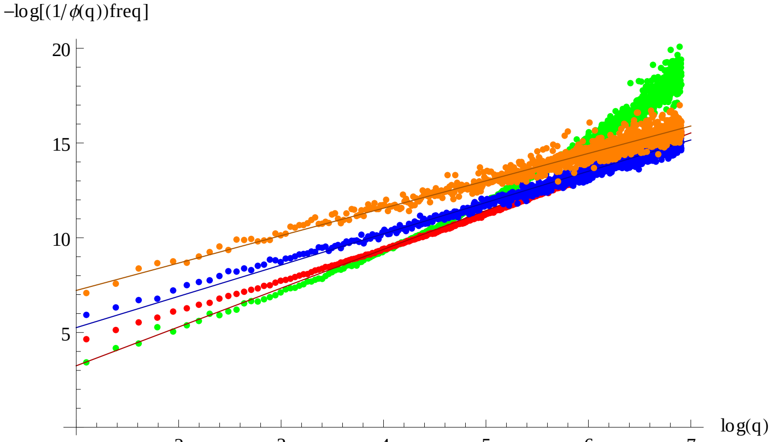
Histogram of scl on $\phi(a+b+c+ABC)$ for

$$\phi \circ \iota : F_3 \to F_2$$

where $\iota : F_3 \to F_2$ is inclusion as an index 2 subgroup, and $\phi \in \mathrm{Out}(F_3)$:



Plot of $\log(q)$ versus $-\log(\text{freq}(q)/\phi(q))$ where $\text{scl} = p/q$ on random words of length 30, 40, 50, 60 in rank 2:



The scl spectrum of a group G is the set

$$\mathrm{spec}(G) := \{\mathrm{scl}(w) \mid w \in [G, G]\}$$

$\mathrm{spec}(F)$ is the same for all free groups F (of rank at least 2).

For a free group F:

1. $\mathrm{spec}(F)$ is a subset of $\mathbb{Q}$.
2. The least positive number in $\mathrm{spec}(F)$ is $1/2$.
3. $\mathrm{spec}(F)$ contains a well-ordered subset of order type ω^ω.
4. For every rational number p/q there is some integer n so that $p/q + n \in \mathrm{spec}(F)$.

Question: is $1/2$ an isolated value in $\mathrm{spec}(F)$?

Question: is $\mathrm{spec}(F)$ dense in any interval?

Question: does $\mathrm{spec}(F) = 0 \cup \mathbb{Q} \cap [1/2, \infty)$?

3. Dynamics

scl measures the failure of G to be commutative by mapping commutators into G.

There is a "dual" method to measure the failure of G to be commutative, by looking at functions from G to $\mathbb{R}$ which fail to be homomorphisms.

Definition: If G is a group, a function $\phi : G \to \mathbb{R}$ is a *homogeneous quasimorphism* if $\phi(g^n) = n\,\phi(g)$, and for all g, h in G,

$$|\phi(gh) - \phi(g) - \phi(h)| \le D(\phi)$$

for some least D, called the *defect*. The space of homogeneous quasimorphisms on G is denoted $Q(G)$.

Quasimorphisms are how scl makes contact with geometry and dynamics.

The quotient Q/H^1 is a Banach space with the norm $2D$; it is the isometric dual of B^H with its scl norm.

Example: The group $\mathrm{Homeo}^+(S^1)$ has a universal central extension

$$\mathbb{Z} \to \mathrm{Homeo}^+(S^1)^\sim \to \mathrm{Homeo}^+(S^1)$$

Elements of $\mathrm{Homeo}^+(S^1)^\sim$ are homeomorphisms of $\mathbb{R}$ commuting with integer translation.

Poincaré's *rotation number* $\mathrm{rot} : \mathrm{Homeo}^+(S^1)^\sim \to \mathbb{R}$ is defined by

$$\mathrm{rot}(g) := \lim_{n\to\infty} \frac{g^n(0)}{n}$$

It is a homogeneous quasimorphism with $D(\mathrm{rot}) = 1$.

Example: There is a similar central extension

$$\mathbb{Z} \to \mathsf{Sp}(2n, \mathbb{R})^{\sim} \to \mathsf{Sp}(2n, \mathbb{R})$$

and a *symplectic rotation number* $\mathrm{rot}_n : \mathsf{Sp}(2n, \mathbb{R})^{\sim} \to \mathbb{R}$.

In place of the circle S^1, one uses the action on Λ_n, the space of Lagrangian planes in $\mathbb{R}^{2n}$. This has $\pi_1(\Lambda_n) = \mathbb{Z}$.

$\Lambda_1 = S^1$; Λ_2 is a non-orientable S^2 bundle over S^1.

$\mathrm{rot}_n(g)$ measures how fast $g^n(\pi)$ "winds around" Λ_n, for some $\pi \in \Lambda_n$.

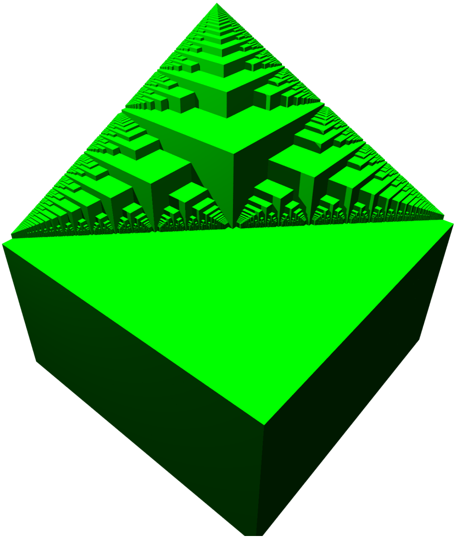
Given $\phi : F_2 \to \mathrm{Homeo}^+(S^1)^\sim$ with $\mathrm{rot}(a) = r$ and $\mathrm{rot}(b) = s$.

If rot were a homomorphism, we would have $\mathrm{rot}(ab) = r + s$.

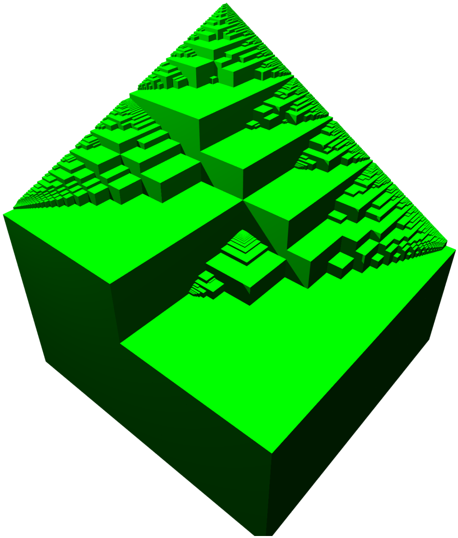
Question: Given the value of $\mathrm{rot}(a)$ and $\mathrm{rot}(b)$, what are the possible values of $\mathrm{rot}(ab)$?

Question: For any fixed $w \in F_2$, given the value of $\mathrm{rot}(a)$ and $\mathrm{rot}(b)$, what are the possible values of $\mathrm{rot}(w)$? What is the maximum value R?

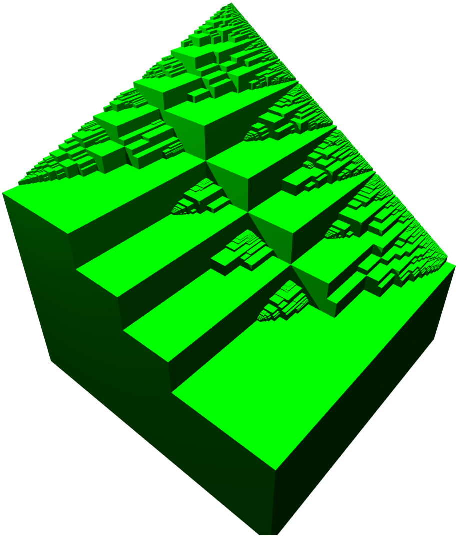
Example: The ziggurat (i.e. the graph of R) for $w = ab$.



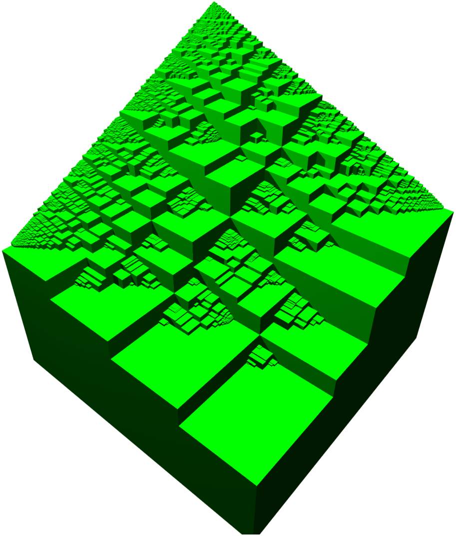
Example: The ziggurat for $w = abaab$.



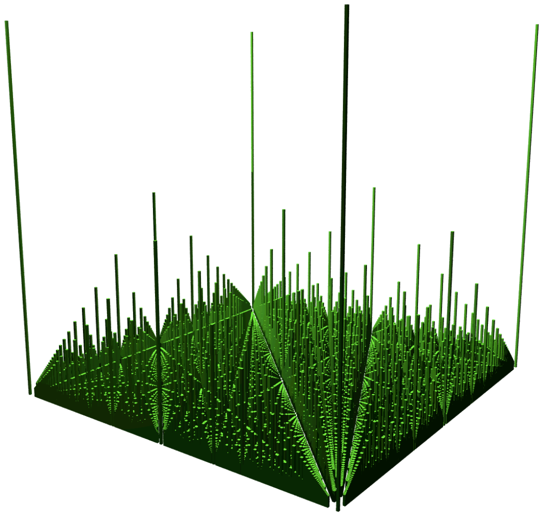
Example: The ziggurat for $w = abaabaaab$.



Example: The ziggurat for $w = abbbabaaaabbabb$.



Example: The "ziggurat" for $w = abAB$.



1. Greedy rationals!
2. Polyhedra!
3. Power laws!
4. Arnold tongues (KAM)??
5. Nonlinear phase locking (spontaneous synchronization)???

?????????

Thank you for your attention!

References

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  `github.com/aldenwalker/scallop`