The topological Cauchy–Schwarz inequality

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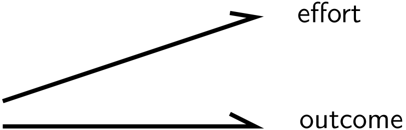
Question: Why is it important for a professional mathematician to have an intimate familiarity with “elementary” mathematics? Isn’t all that stuff better left to computers or engineers?

Answer: Because the fundamental patterns that underlie elementary mathematics are the templates for the byzantine intricacies of current research.

Corollary: It is worthwhile to revisit this elementary material from a professional perspective, and to try to reimagine it in different ways.

The Cauchy–Schwarz inequality says (roughly) that the greatest efficiency is achieved when efforts are aligned with outcomes.

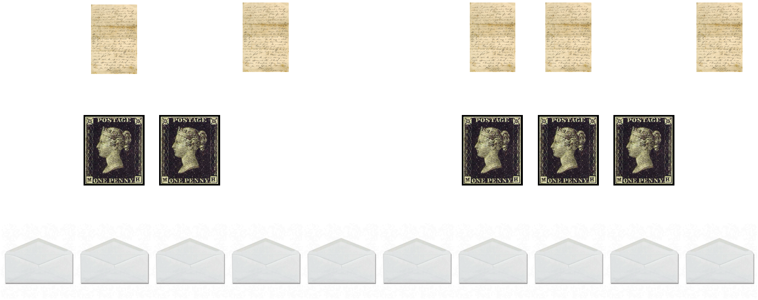
Augustin-Louis Cauchy (1789-1857) and Hermann Amandus Schwarz (1843-1921)



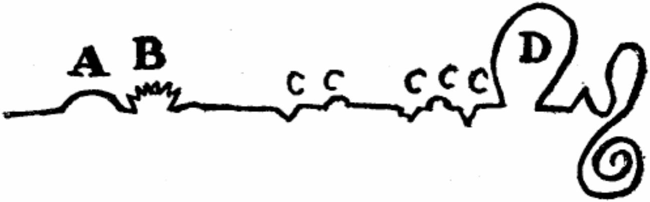
Example: A horse pulling a barge down a river is most efficient when the force exerted by the horse is in the direction of motion of the barge (i.e. it is aligned with the direction of the river).

The last horse-drawn barge on the River Lea — in 1955 near Waltham Abbey

Example: Alice stuffs flyers in envelopes, while Bob attaches the stamps. If Alice only stuffs flyers in half of the envelopes, and Bob only puts stamps on half of the envelopes, the most flyers will reach their destination if they choose the same half.

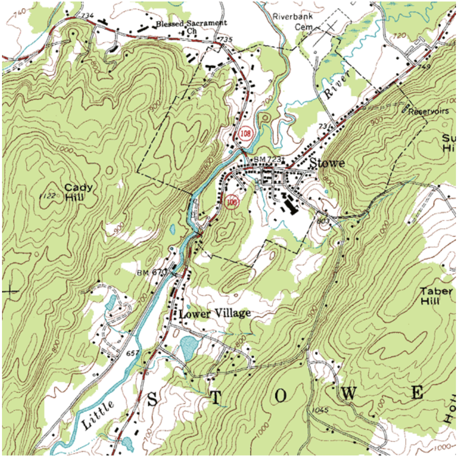


Example: The shortest path between two points is a straight line.



Lawrence Sterne, *Tristram Shandy*

Example: To descend a mountain as quickly as possible, always move perpendicularly to the contour lines on a topographic map.



USGS topographic map of Stowe, Vermont, USA

More precisely, the Cauchy-Schwarz inequality is a statement about vectors. A vector is a list of numbers, often written vertically. If we have two vectors A, B (i.e. lists) with the same number of entries, we can multiply them entry by entry, and add up the result; this is the inner product of A and B, denoted $\langle A, B \rangle$. For example:

$$A := \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} \qquad B := \begin{pmatrix} 2 \\ 4 \\ 3 \\ -7 \end{pmatrix}$$

$$\left.\begin{array}{c} 1 \times 2 = 2 \\ 3 \times 4 = 12 \\ 2 \times 3 = 6 \\ 4 \times -7 = -28 \end{array}\right\} \text{ so } \langle A, B \rangle \text{ is } 2 + 12 + 6 + (-28) = -8$$

The Cauchy–Schwarz inequality says that for two vectors A and B,

$$\langle A, B\rangle \leq \text{the average of } \langle A, A\rangle \text{ and } \langle B, B\rangle$$

with equality if and only if A and B are proportional (i.e. one can be obtained from the other by scaling the entries by a fixed positive number).

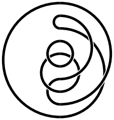
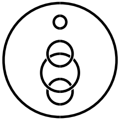
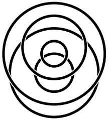
Warning: "average" here means geometric mean; the geometric mean of α and β is $\sqrt{\alpha \times \beta}$. It is often safe to use "max" in place of "average".

With the C–S inequality as a template, we can look for generalizations.

⟨ ,  ⟩ ≤ max ⟨ ,  ⟩, ⟨ ,  ⟩

Example: Look at matchings of finite sets of points. For example,

if $A :=$  and $B :=$  ,

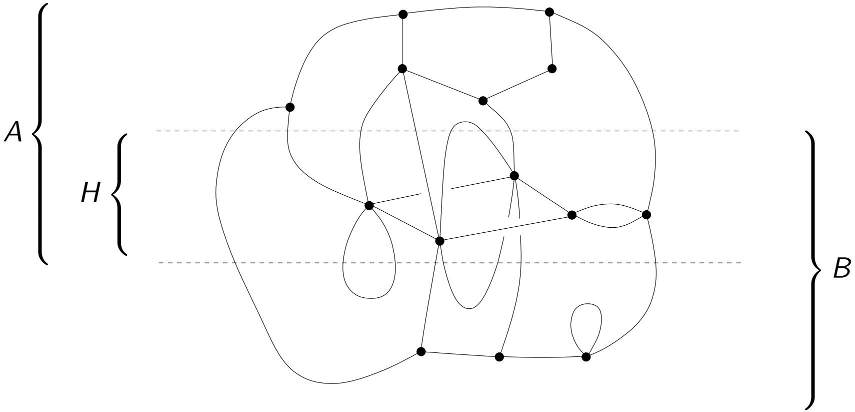
then $AB =$ , $AA =$ , and $BB =$ .

If $c(\cdot)$ means number of components, then

$$c(AB) \leq \max c(AA), c(BB)$$

with equality if and only if $A = B$.

Example: Let H be a fixed finite graph, and let $\mathcal{N}_H$ denote the set of finite graphs containing a copy of H as a subgraph (up to isomorphism).



If A and B are in $\mathcal{N}_H$, we can glue them together along their copies of H to produce AB.

Theorem (Calegari-Freedman-Walker)
For any finite graph H there is a complexity function c defined on the set of all finite graphs so that for any $A, B \in \mathcal{N}_H$

$$c(AB) \leq \max c(AA), c(BB)$$

with equality if and only if $A = B$.

The function c in the proof of this theorem is complicated (and is not obviously of independent interest to graph theorists).

Question: Is there a meaningful function c satisfying the C–S axiom?

One motivation to study such questions comes from quantum mechanics.

The quantum state of a system is described by a vector A in a vector space.

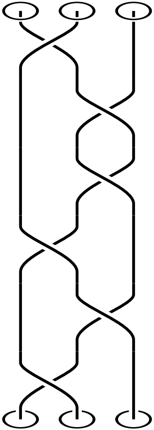
If we perform a suitable experiment, the probability amplitude of finding the system in some (possibly different) state B is the value of the inner product $\langle A, B \rangle$.

The Cauchy–Schwarz inequality ensures that the probability of any outcome is always between 0 and 1.

One nice example might have applications to quantum computing. A topological quantum computer could be a superfluid of fermionic cold atoms in a macroscopic quantum state known as a topological phase of matter. In such a state, physical properties that can be measured depend only on topology.

Vortices in the fluid can be adiabatically (i.e. slowly) braided around each other; the topological pattern of the braiding determines the computation.

Q: Why is a raven like a writing desk?



A: This margin is too small for the answer.

The topological stability of these vortices (also called anyons) could potentially solve one of the most difficult practical issues with “conventional” quantum computers, namely the problem of error correction.

This proposal is being pursued by Alexei Kitaev, Mike Freedman, Kevin Walker, Chetan Nayak and others at Microsoft Station Q.

There are two ways to look at a computer. One way is to look at the computation to be performed and ask what the answer will be. The other way is to look at the answer, and ask what computation could have led to it.

We take the latter perspective, and are interested in what kinds of information a topological quantum computer is sensitive to (in principle).

Basic Question: How much topology could such a computer see? Is quantum mechanics powerful enough to distinguish braids?

To a topologist, this problem can be abstracted as follows.

We fix a manifold S of dimension d, and let $\mathcal{M}_S$ denote the complex vector space spanned by the set of all $(d+1)$-dimensional manifolds bounding S (up to isomorphism).

There is a (Hermitian) pairing on $\mathcal{M}_S$, with values in the complex vector space spanned by the set of all closed $(d+1)$-dimensional manifolds, obtained by gluing manifolds along S, and pairing coefficients.

Manifold Positivity Question: Is there a nonzero vector $v \in \mathcal{M}_S$ so that $\langle v, v \rangle = 0$?

If there is no such vector we say Manifold Positivity holds (for a given value of d).

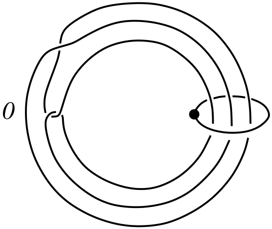
Theorem(Freedman-Kitaev-Nayak-Slingerland-Walker-Wang,Kreck-Teichner)

Manifold Positivity holds for $d = 0$ or 1.

Manifold Positivity fails for all $d \geq 3$.

(what about $d = 2$?)

Example: M is the Mazur manifold; the following surgery diagram denotes both a 3-manifold S and the 4-manifold M with $\partial M = S$.



There is an involution $\theta : S \to S$ which does not extend over M, so M and $\theta(M)$ are distinct elements of $\mathcal{M}_S$. But

$$MM = M\theta(M) = \theta(M)M = \theta(M)\theta(M) = S^4$$

so $v = M - \theta(M) \in \mathcal{M}_S$ is nonzero, with $\langle v, v \rangle = 0$.

Topological Cauchy–Schwarz inequality (Calegari-Freedman-Walker)
There is a complexity function c defined on all closed 3-manifolds, so that if S is any surface, and A, B are any two 3-manifolds with $\partial A = \partial B = S$, then

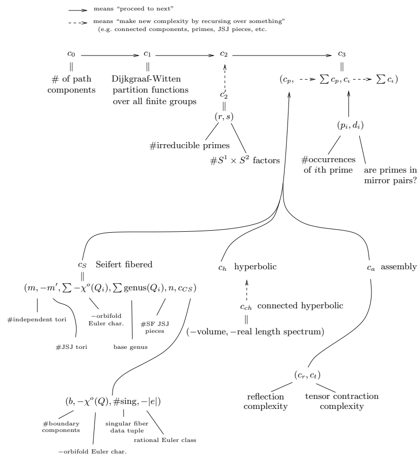
$$c(AB) \leq \max c(AA), c(BB)$$

with equality if and only if $A = B$.

Corollary: Manifold Positivity holds for $d = 2$.

Proof: Let $v = \sum_i \xi_i M_i$ so $\langle v, v \rangle = \sum_{i,j} \xi_i \overline{\xi_j} M_i M_j$. Let N be the 3-manifold maximizing $c(M_i M_j)$ in this collection. Then the coefficient of N is of the form $\sum_{i_k} \xi_{i_k} \overline{\xi_{i_k}} = \sum_{i_k} |\xi_{i_k}|^2 > 0$.

The definition of the complexity function c is recursive, and depends on the hierarchical classification theorem for 3-manifolds.

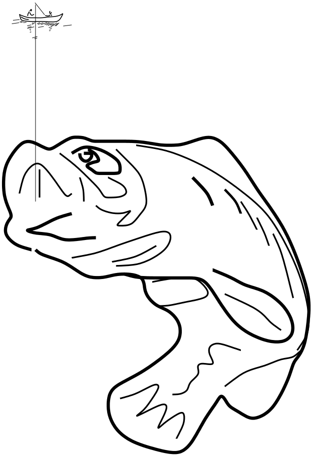


Question: Why is it that “simple” ideas like the Cauchy–Schwarz inequality have such apparently boundless potential?

One hypothesis is that mathematical theories, like pearls in oysters, develop by the gradual accretion of layers around some initial nugget, itself of no great intrinsic value.

Another hypothesis says that our psychological access to the mathematical world is indirect.

A good mathematical idea is a slender but strong thread extending down into the unseen depth of reality, which can be slowly reeled in, maybe over centuries.



(I)gnorance, which simplifies and clarifies, which selects and omits, with a placid perfection unattainable by the highest art $\cdots$ (one must) row out over that great ocean of material, and lower down into it, here and there, a little bucket, which will bring up to the light of day some characteristic specimen, from those far depths, to be examined with a careful curiosity.

Lytton Strachey, *Eminent Victorians*

The End

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