HYPERBOLIC 3-MANIFOLDS, TAMENESS, AND AHLFORS' MEASURE CONJECTURE

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ABSTRACT. This is a series of lecture notes giving an exposition of the proof of Ahlfors' Measure Conjecture. They correspond roughly to the contents of a mini-course given at the Independent University of Moscow during February 19-23, 2007, at the conference entitled "laminations and group actions in dynamics".

1. INTRODUCTION

The purpose of these notes is to give a brief outline of the proof of Ahlfors' measure conjecture, assuming as few prerequisites as possible, following the arguments of [7]. Obviously such a short summary cannot hope to be completely self-contained, but at least I hope these notes give an overview of the logical structure of the argument, and briefly touch on the main points along the way, while indicating how the various pieces fit into the general theory of Kleinian groups. For instance, our survey of the geometry of simply degenerate ends, and the Ending Lamination Conjecture in particular, is very sketchy; a much more substantial overview of this topic can be found in the notes by Minsky [24].

General background on Kleinian groups can be found in Thurston's notes [35] and in Curt McMullen's notes [22].

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2. LECTURE 1 — HYPERBOLIC 3-MANIFOLDS

Basic references for 3-manifold topology are Hempel's book [15] and Hatcher's online notes [14]. Basic references for the theory of Kleinian groups are Thurston's notes [35] and the book by Matsuzaki and Taniguchi [20].

2.1. Kleinian groups.

Let $\mathbb{B}$ denote the open unit ball in $\mathbb{R}^3$.

Definition 2.1.1. A *Kleinian group* Γ is a finitely generated discrete group of conformal (i.e. angle-preserving) automorphisms of $\mathbb{B}$.

We denote the closure of $\mathbb{B}$ by $\overline{\mathbb{B}}$, and we denote its boundary by S^2_∞.

A conformal automorphism of $\mathbb{B}$ extends uniquely to a conformal automorphism of $\mathbb{R}^3 \cup \infty$, and restricts to a conformal automorphism of S^2_∞. Conversely, any conformal automorphism of S^2_∞ extends to $\mathbb{B}$.

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The stabilizer in $\mathrm{Conf}(\mathbb{B})$ of a point $p \in \mathbb{B}$ is compact, and isomorphic as a group to $O(3)$. It follows that $\mathrm{Conf}(\mathbb{B})$ preserves an isotropic Riemannian metric on $\mathbb{B}$, unique up to scaling. If we normalize to have curvature -1 we obtain a model of *hyperbolic* 3*-space* which we denote $\mathbb{H}^3$, and we have identifications

$$\mathrm{Isom}(\mathbb{H}^3) = \mathrm{Conf}(\mathbb{B}) = \mathrm{Conf}(S^2_\infty)$$

If we think of S^2_∞ as the Riemann sphere $\mathbb{CP}^1$, the subgroup of orientation-preserving maps can be identified with the matrix group $\mathrm{PSL}(2,\mathbb{C})$.

Elements of $\mathrm{PSL}(2,\mathbb{C})$ come in three kinds:

(1) Elliptic elements, which fix an axis pointwise and rotate space around it
(2) Hyperbolic (loxodromic) elements, which translate along an axis, possibly with a screw motion
(3) Parabolic elements, which act on $\mathbb{CP}^1$ conjugate to a translation $z \to z + w$

Elliptic elements are distinguished by having fixed points in $\mathbb{B}$. In a Kleinian group, the elliptic elements are those of finite order. Hyperbolic elements have two fixed points in S^2_∞, and their axis is the infinite geodesic joining them. Parabolic elements have a unique fixed point in S^2_∞.

Example 2.1.2. Let P be a hyperbolic polyhedron with finitely many sides, and dihedral angles of the form $\frac{\pi}{n_i}$ for various integers n_i. The group generated by reflections in the sides of P is discrete, and is therefore a Kleinian group. For example, Poincaré showed that there is a hyperbolic triangle with dihedral angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

Example 2.1.3 (Andreev's Theorem). Let P be a combinatorial polyhedron. That is, the underlying topological space of P is a closed 3-ball, and ∂P contains a finite trivalent graph whose complementary regions are all combinatorial disks. Equivalently, P is given by the data of Γ, a connected trivalent graph Γ in S^2. Suppose P is not a combinatorial tetrahedron or triangular prism. Suppose further that if Γ' denotes the dual graph of Γ (in S^2) then every cycle in Γ' of length 3 surrounds a vertex of P, and every cycle of length 4 surrounds an edge. Then Andreev [4] showed that P can be realized in $\mathbb{H}^3$ as a polyhedron with totally geodesic sides, and all dihedral angles equal to $\pi/2$.

It follows that the group generated by reflections in any subset of the sides of P is a Kleinian group, as in Example 2.1.2. Also see [35], especially §13.6 for a generalization of Andreev's theorem, and a nice conceptual proof involving circle packings.

Example 2.1.4 (Schottky group). Let C_i be the boundary circles of a finite collection of disjoint closed round disks in $\mathbb{C} \subset \mathbb{CP}^1 = S^2_\infty$. The group Γ generated by inversions in the circles is a Kleinian group. If Γ^+ is the subgroup of Γ consisting of orientation-preserving isometries, then Γ^+ is a free group, by Klein's ping-pong Lemma (see e.g. [13]). The group Γ^+ is an example of what is known as a *Schottky group*.

Example 2.1.5. Let A be the ring of integers in the field $\mathbb{Q}(\sqrt{-3})$. Then $\mathrm{PSL}(2,A)$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and is finitely generated. In fact, it turns out that this group has index 12 in the two-generator group $\pi_1(S^3 - K)$ where K is the figure 8 knot. More generally, it is possible to construct many discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ (so-called *arithmetic* Kleinian groups) using number theoretic methods.

2.2. **Torsion elements.** Since Γ acts discretely on $\mathbb{H}^3$ by isometries, the quotient $\mathbb{H}^3/\Gamma$ is a hyperbolic orbifold. If Γ is torsion free, the quotient is a manifold.

In practice, one can ignore torsion elements by means of the following theorem:

Theorem 2.2.1 (Selberg's Lemma [30])**.** *Let Γ be a Kleinian group. Then Γ contains a torsion-free finite index subgroup Γ'.*

The proof has nothing to do with discreteness, and is a general fact about any finitely generated matrix group Γ. Basically, one notes that the matrix entries lie in a finite algebraic extension of $\mathbb{Q}$. Reducing modulo various primes gives surjective homomorphisms from Γ to many finite groups; Γ' is the intersection of the kernel of various such surjections. (Note that though $\mathrm{PSL}(2,\mathbb{C})$ is not itself a matrix group, it is isomorphic to the matrix group $\mathrm{SO}(3,1) \subset \mathrm{GL}(4,\mathbb{R})$.)

In what follows, we will typically restrict attention to torsion-free groups.

2.3. **Lattices and thick-thin.** A Kleinian group Γ is a *lattice* if the quotient $M = \mathbb{H}^3/\Gamma$ has finite volume. It is a *uniform lattice* if M is compact, and *non-uniform* otherwise. The property of being a lattice is preserved after passing to a finite index subgroup, so we consider the case that M is a manifold.

To understand the geometry of a finite volume hyperbolic manifold, one studies the *thick-thin* decomposition. Fix a small $\epsilon > 0$, let M_ϵ (the *thin* part) be the subset of M for which the injectivity radius is $< \epsilon$, and let $M_{\geq\epsilon}$ (the *thick* part) be its complement. Of course, this decomposition makes sense whether M has finite volume or not. The difference is that if M has finite volume, $M_{\geq\epsilon}$ is compact.

The thin part can be understood using Margulis' Lemma, which in this context says the following:

Theorem 2.3.1 (Margulis' Lemma [19])**.** *Let Γ be a torsion-free orientation-preserving Kleinian group, with quotient $M = \mathbb{H}^3/\Gamma$. There is a universal constant ϵ (in each dimension) such that every component of M_ϵ is either an open solid torus neighborhood of a simple closed geodesic (a "Margulis tube"), or the quotient of an open horoball by a free parabolic $\mathbb{Z}$ or $\mathbb{Z}^2$ action (a "$\mathbb{Z}$-cusp" or a "$\mathbb{Z}^2$-cusp").*

The constant ϵ is usually called a *Margulis constant*.

Margulis' Lemma actually says that if $p \in \mathbb{H}^3$ and Γ_p is the subgroup of a Kleinian group generated by elements which move p less than ϵ, then Γ_p contains a finite index subgroup which is nilpotent. The conclusion in the statement of the theorem follows by an analysis of the nilpotent subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

Margulis' Lemma is really a combination of two observations. First, observe that $[\cdot,\cdot]$ is constant on $G \times \mathrm{id}$ and $\mathrm{id} \times G$, and therefore $d[\cdot,\cdot]$ is the zero map at $\mathrm{id} \times \mathrm{id}$ in a Lie group G. In particular, in a discrete subgroup of a Lie group, a subgroup generated by elements sufficiently close to the identity is nilpotent. Second, the point stabilizers in $\mathbb{H}^3$ are compact, and therefore an element of $\mathrm{PSL}(2,\mathbb{C})$ which moves a point a small distance has a small power which is close to id in $\mathrm{PSL}(2,\mathbb{C})$.

If Γ is a lattice, $M_{\geq\epsilon}$ is a compact manifold with finitely many boundary components. By the lemma, these components are all tori, which either bound Margulis tubes, or else $\mathbb{Z}^2$ cusps. A $\mathbb{Z}^2$ cusp is topologically a product $T \times (0, \infty)$ so M is homeomorphic to the interior of a compact manifold, whose boundary components are a finite collection of tori. Notice that if Γ contains no parabolic elements, then M is necessarily closed. In particular, a lattice is uniform if and only if it contains no parabolic elements. In the sequel, we will typically restrict attention to parabolic-free groups.

The geometry of lattices is very constrained. The fundamental theorem in this area is the theorem of Mostow-Prasad. This is a theorem which applies quite generally in all dimensions ≥ 3; we state it here just for dimension 3.

Theorem 2.3.2 (Mostow-Prasad Rigidity [27])**.** *Suppose M_1, M_2 are complete, finite volume hyperbolic 3 manifolds. Every isomorphism of fundamental groups $f_* : \pi_1(M_1) \to \pi_1(M_2)$ is induced by a unique isometry from M_1 to M_2.*

This theorem says that if Γ_1, Γ_2 are lattices which are isomorphic as abstract *groups*, then they are isomorphic as Kleinian groups; i.e. they are conjugate as subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

Remark 2.3.3. Mostow's proof addresses the case that the Γ_i are uniform; the potentially non-uniform case is treated by Prasad.

2.4. **Limit sets.** Let Γ be a Kleinian group. Since Γ is discrete, points in $\mathbb{B}$ have finite stabilizers and discrete orbits under Γ. But the orbit Γp of $p \in \mathbb{B}$ will typically have accumulation points in S^2_∞.

Definition 2.4.1. Let Γ be a Kleinian group. The *limit set* of Γ (denoted $\Lambda(\Gamma)$) is the set of accumulation points of an orbit Γp. That is,

$$\Lambda(\Gamma) := \overline{\Gamma p} \cap S^2_\infty \subset \overline{\mathbb{B}}$$

From the definition we see that Λ is closed and Γ-invariant.

Suppose s is a limit point, so that there are $\gamma_i \in \Gamma$ for which $\gamma_i(p) \to s$. Since S^2_∞ is compact, there is $t \in S^2_\infty$ for which $\gamma_i^{-1}(p) \to t$ for some subsequence (here it is possible that $t = s$). It turns out that this subsequence converges uniformly on compact subsets of $\overline{\mathbb{B}} - t$ to the constant map sending every point to s. If $t = s$ for every subsequence, this is trivially true; otherwise it follows by Montel's theorem and the fact that Γ is discrete in $\mathrm{PSL}(2, \mathbb{C})$.

As a consequence we see that the definition of $\Lambda(\Gamma)$ does not depend on the choice of the points p. Moreover, if Γ' is finite index in Γ, then clearly $\Lambda(\Gamma') = \Lambda(\Gamma)$.

Moreover, we easily deduce the following facts:

Theorem 2.4.2. *Let Λ be the limit set of Γ. Suppose Λ contains at least three points. Then*

(1) *Λ is the unique minimal, closed, nonempty, Γ-invariant subset of S^2_∞.*
(2) *Λ is perfect.*
(3) *Γ acts properly discontinuously on $S^2_\infty - \Lambda$.*

If Λ has ≤ 2 points, we say Γ is *elementary*. A Kleinian group is elementary if and only if it has a finite index subgroup which is abelian. In the sequel we will typically restrict attention to non-elementary Kleinian groups.

Definition 2.4.3. The *domain of discontinuity*, denoted $\Omega(\Gamma)$, is the complement

$$\Omega(\Gamma) := S^2_\infty - \Lambda(\Gamma)$$

If Γ is understood, we abbreviate these sets by Λ and Ω respectively. Note that Ω is open in S^2_∞, and Γ acts properly discontinuously on Ω. Since Γ acts conformally on S^2_∞, the quotient $S = \Omega/\Gamma$ is a surface orbifold with a well-defined conformal structure. If Γ is torsion-free, S is a conformal surface; if further Γ is orientation preserving, S has the natural structure of a Riemann surface.

Notice that the partition $S^2_\infty = \Lambda \cup \Omega$ is natural with respect to the Γ action, at least when Γ is non-elementary: Γ acts minimally on Λ, and properly discontinuously on Ω. All the interesting dynamics of Γ, therefore, lies in Λ, and it becomes important to understand basic properties of Λ.

One of the most basic questions about limit sets was Ahlfors' so-called Measure Conjecture:

Conjecture 2.4.4 (Ahlfors' Measure Conjecture [2])**.** *Let Γ be a Kleinian group. Then the limit set Λ is either all of S^2, or has (Lebesgue) measure zero.*

We will see that this conjecture is true, and follows from a topological theorem — Marden's Tameness Conjecture (now a theorem) — together with work of Thurston, Bonahon and Canary.

2.5. **Quasiconformal deformations.** A conformal structure on a vector space is a definite inner product up to scale. In two dimensions, this data is determined (with respect to some reference basis) by prescribing the direction of the major axis of the unit ellipse, and its eccentricity. On a 2-sphere, this data may be prescribed pointwise and measurably, where it defines what is said to be a *measurable conformal structure*.

Measurable conformal structures in two dimensions can be understood completely, using the following theorem:

Theorem 2.5.1 (Ahlfors-Bers Measurable Riemann Mapping Theorem [3])**.** *Any measurable conformal structure on S^2 is conjugate (by a homeomorphism which is unique up to post-composition with an element of* $\mathrm{PSL}(2,\mathbb{C})$*) to the standard conformal structure on* $\mathbb{CP}^1$.

Technically, the data takes the form of a *Beltrami differential*, which is a measurable complex-valued differential $\mu := \mu(z)\frac{d\bar{z}}{dz}$ on $\mathbb{CP}^1$ for which $\sup|\mu| < 1$, and the measurable Riemann mapping theorem guarantees the existence of a *quasiconformal map* $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ (i.e. a map which distorts the conformal structure a bounded amount), unique up to post-composition with an element of $\mathrm{PSL}(2,\mathbb{C})$, which satisfies

$$\frac{\bar{\partial} f}{\partial f} = \mu$$

in the sense of distribution.

Remark 2.5.2. Theorem 2.5.1 as stated was actually first proved by Morrey [26]. The contribution of Ahlfors-Bers was to establish the holomorphic dependence of f on the parameter μ. However the terminology is well-established.

If Γ is a Kleinian group, and μ is a Beltrami differential on $\mathbb{CP}^1$ which is invariant under Γ, the uniformizing map f defines a quasiconformal conjugacy between Γ and the Kleinian group $\Gamma_f := f\Gamma f^{-1}$. The group Γ_f is called a *quasiconformal deformation* of Γ.

The groups Γ and Γ_f, although abstractly isomorphic, are typically not conjugate in $\mathrm{PSL}(2,\mathbb{C})$. The set of representations of the abstract group Γ into $\mathrm{PSL}(2,\mathbb{C})$ up to conjugacy is an algebraic variety, and its tangent space at the identity is parameterized by a certain cohomology group, namely $H^1(\Gamma;\mathfrak{sl}(2,\mathbb{C}))$. Therefore the tangent to a family of quasiconformal deformations determines an element of $H^1(\Gamma;\mathfrak{sl}(2,\mathbb{C}))$. Since Γ is finitely generated, this cohomology group is *finite dimensional*.

If Γ and Γ_f are actually conjugate in $\mathrm{PSL}(2,\mathbb{C})$, this conjugacy necessarily takes fixed points of Γ to corresponding fixed points of Γ_f, and is therefore determined pointwise on Λ. So in this case, μ must vanish a.e. on Λ. If the action of Γ on Λ had a dissipative part (i.e. a "fundamental domain" $F \subset \Lambda$ of positive measure which is disjoint from its Γ-translates) one could construct an infinite dimensional family of Γ-invariant Beltrami differentials supported on the orbit $\Gamma \cdot F$, which would inject into the finite dimensional cohomology group $H^1(\Gamma;\mathfrak{sl}(2,\mathbb{C}))$; this contradiction shows that the action of a Kleinian group on its limit set is *conservative* (i.e. not dissipative).

Sullivan shows in general that a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ (with no hypothesis that it be finitely generated) admits no nonzero invariant Beltrami differentials on a subset of S^2_∞ where it acts conservatively. Roughly speaking, if μ is a Γ-invariant differential on S^2_∞, we "zoom in" near a point of density for the support of μ until μ looks almost constant. Where Γ acts conservatively, there are lots of elements with Jacobian very close to 1 on most of the measure of Γ. For, otherwise, there are lots of elements which are uniformly expanding on most of the conservative set, which is absurd, since its total measure is finite. So after zooming in, we obtain many elements of Γ whose 1-jet is very close to the identity, and which preserve an almost constant (nonzero) differential. The set of such elements is a normal family, so one obtains nontrivial limits of infinite sequences in Γ, violating discreteness.

Hence we have

Theorem 2.5.3 (Sullivan, no invariant line fields [33])**.** *Let Γ be a Kleinian group. Then Λ does not support any Γ-invariant Beltrami differentials.*

Quasiconformal deformations are therefore supported on the domain of discontinuity Ω. Obviously, if Ω/Γ and $f(\Omega)/\Gamma_f$ are different as (marked) conformal structures, then Γ and Γ_f are not conformally conjugate, and are distinct *as Kleinian groups*.

Conversely, any two (marked) conformal structures on Ω/Γ are related by a Γ-invariant Beltrami differential. The space of (marked) conformal structures on a Riemann surface S is called the *Teichmüller space* of S, and denoted $\mathcal{T}(S)$.

So we have:

Theorem 2.5.4. *Let Γ be a Kleinian group. The space of quasiconformal deformations of Γ is parameterized by the Teichmüller space $\mathcal{T}(\Omega/\Gamma)$.*

When S is of finite type (i.e. conformally equivalent to a closed surface of genus g with n points removed), and $\chi(S)$ is negative, $\mathcal{T}(S)$ has complex dimension $3g - 3 + n$. In general, the dimension may be computed via Riemann-Roch. When S is not of finite type, $\mathcal{T}(S)$ is infinite dimensional.

Note that the tangent space to $\mathcal{T}(\Omega/\Gamma)$ can be identified with the finite dimensional space $H^1(\Gamma;\mathfrak{sl}(2,\mathbb{C}))$. It follows that every component of Ω/Γ is of finite type.

Example 2.5.5 (Quasifuchsian groups). Let $C_1,\dots,C_n$ be a union of round circles in the plane with disjoint interiors, which are tangent to each other in a ring around a round circle C to which they are all orthogonal. Inversion in each C_i preserves C, and the Kleinian group Γ generated by these reflections contains a (punctured) surface group Γ^+ with index 2. The limit sets of Γ and Γ^+ are both equal to C. As we slide the C_i around in the plane, keeping them tangent in the same combinatorial pattern, the limit set C deforms into a *quasicircle*. When two circles bump into each other, the Kleinian group Γ_t^+ obtained by deforming Γ^+ develops an "accidental parabolic", and the quasicircle pinches off into a cactus; see Figure 1.

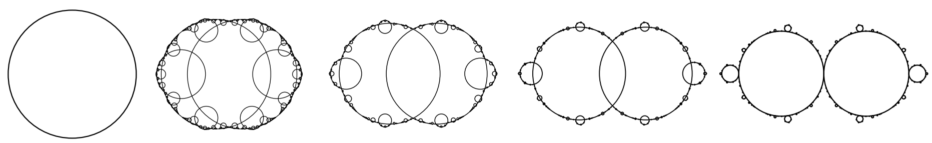

FIGURE 1. As the group is deformed, the limit set degenerates from a circle into a cactus.

Example 2.5.6 (Simultaneous uniformization). Let S be a closed Riemann surface, and let $\overline{S}$ denote S with the opposite orientation. Let $t \in \mathcal{T}(S)$ be a point in Teichmüller space. Identifying the universal cover of S with a totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^3$ (by the usual uniformization theorem) determines a discrete faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ whose image stabilizes a round circle. So Ω consists of two open disks $\Omega = H \cup \overline{H}$, and $\Omega/\rho(\pi_1(S))$ is a union of two copies of S (one with opposite orientation) with (marked) conformal structures corresponding to $t \in \mathcal{T}(S)$ and $\overline{t} \in \mathcal{T}(\overline{S})$. Let $t' \in \mathcal{T}(S)$ be another marked conformal structure on S, determining a Riemann surface S' and a homeomorphism $f : S \to S'$. A reasonable choice of map f determines a Beltrami differential on S by $\mu = \overline{\partial} f/\partial f$, which lifts to a $\pi_1(S)$-invariant Beltrami differential on $H \subset \Omega$. Extend this Beltrami differential to all of $\mathbb{CP}^1$ by setting it equal to 0 outside H. Then apply Theorem 2.5.1 to obtain a new representation $\rho' : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ which is quasiconformally conjugate to ρ, for which $\Omega/\rho'(\pi_1(S))$ represents $(t',\overline{t}) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$.

Theorem 2.5.3 can be used to give a proof of the uniform case of Mostow's Rigidity Theorem (i.e. Theorem 2.3.2), as follows. An abstract isomorphism between Kleinian groups Γ_1 and Γ_2 induces an isometry of their Cayley graphs. The map from Γ_i to an orbit determines a quasi-isometry (i.e. a "coarse" map which is bilipschitz on a large scale) between their respective Cayley graphs and $\mathbb{H}^3$, and therefore a quasi-isometry from $\mathbb{H}^3$ to itself. This quasi-isometry extends to a quasiconformal conjugacy between the respective actions of Γ_1 and Γ_2 on S^2_∞, and this quasiconformal conjugacy determines an invariant Beltrami differential. However, since the Γ_i are lattices, their limit sets are equal to all of S^2_∞, and therefore this Beltrami differential is trivial, by Theorem 2.5.3. In particular, the quasiconformal conjugacy is actually conformal, and the groups are isomorphic as Kleinian groups.

3. Lecture 2 — Scott core and convex core

3.1. Convex core.

Definition 3.1.1. Let Γ be a torsion-free Kleinian group with limit set Λ and quotient manifold M. The *convex hull* of Λ, denoted $C(\Lambda)$, is the intersection of the closed half-spaces in $\mathbb{H}^3$ which contain Λ. The *convex core* of M is the quotient $C(\Lambda)/\Gamma \subset M$.

A Kleinian group Γ is said to be *convex cocompact* if $C(M)$ is compact. For example, the quasifuchsian groups constructed in Example 2.5.5 are convex cocompact.

In a hyperbolic manifold, a set K is convex if it contains every geodesic segment joining any two points in K. The convex core $C(M)$ can be characterized as the smallest nonempty convex subset of M. It is equal to the closure of the union of the set of all closed geodesics in M.

By convexity, the nearest point map defines a deformation retraction from $\mathbb{H}^3$ to $C(\Lambda)$ which covers a deformation retraction from M to $C(M)$. This retraction extends continuously to Ω, and identifies the boundary components of $C(M)$ with the components of Ω/Γ. Boundary components of $C(M)$ are *pleated surfaces*; that is, piecewise isometrically embedded hyperbolic surfaces which are pleated (i.e. bent) along a geodesic lamination. The retraction map

$$r : \Omega/\Gamma \to \partial C(M)$$

is not quite conformal in general, but Sullivan [34] showed that there is a *universal* constant K such that r is K-quasiconformal. Epstein and Marden [10] give a detailed proof of Sullivan's theorem, and obtain the (non-optimal) estimate $K \le 67$.

3.2. Ends.

Let Γ be orientation-preserving, torsion-free and parabolic free. Define

$$M := \mathbb{H}^3/\Gamma$$

Then M is a hyperbolic manifold. Since $\mathbb{H}^3$ is a ball, $\pi_1(M) = \Gamma$. It follows that $\pi_1(M)$ is finitely generated.

Theorem 3.2.1 (Scott core Theorem [29])**.** *Let M be an irreducible 3-manifold with $\pi_1(M)$ finitely generated. Then there is a compact 3-manifold $\mathcal{C}$ (a "core") which includes in M in such a way that the inclusion is a homotopy equivalence.*

Remark 3.2.2. Shalen gave an independent proof of Theorem 3.2.1 at about the same time as Scott. Nevertheless, the terminology is well-established.

One should be careful to distinguish between the convex core $C(M)$ which is a geometrical object and is unique, and a Scott core $\mathcal{C}$ which is a topological object, and is not *a priori* unique, even up to isotopy in M.

The *ends* of M can be identified with the components of $M - \mathcal{C}$. Since the inclusion of $\mathcal{C}$ into M is a homotopy equivalence, each end may be homotoped into $\mathcal{C}$. The ends are in 1–1 correspondence with the boundary components of $\mathcal{C}$. Note that since $\mathcal{C}$ is compact, M has only finitely many ends, which we enumerate as $\mathcal{E}_1, \ldots, \mathcal{E}_n$.

Remark 3.2.3. If Γ has parabolic elements, one can find a *relative compact core* for M which intersects the closure of each $\mathbb{Z}$-cusp in an annulus, and intersects the closure of each $\mathbb{Z}^2$-cusp in a torus. We let P denote the union of these annulus and torus pieces. Then the ends of $M_{\geq\epsilon}$ are in 1–1 correspondence with the components of $\partial\mathcal{C} - P$, and again there are finitely many such ends. See [11] for a discussion of relative ends.

Now define

$$N := (\mathbb{H}^3 \cup \Omega)/\Gamma$$

Then N is a (typically noncompact) manifold whose boundary is $\partial N = \Omega/\Gamma$, and M is homeomorphic to the interior of N.

Definition 3.2.4. An end of M is *geometrically finite* if it has a neighborhood which does not intersect the convex core $C(M)$.

Informally, the geometrically finite ends are those which are compactified in N by the components of ∂N. A Kleinian group Γ is *geometrically finite* if all the ends of M are geometrically finite. If Γ does not contain parabolic elements, this is equivalent to N being compact. In particular, if Γ is geometrically finite, then M is homeomorphic to the interior of a compact manifold (in the general case, the compact manifold is obtained from N by gluing in tori and annuli if necessary to close up $\mathbb{Z}^2$ cusps and punctures of ∂N).

A fundamental fact about the conformal type of ∂N is the following *finiteness theorem*, due to Ahlfors [2]:

Theorem 3.2.5 (Ahlfors' finiteness theorem [2])**.** *Let Γ be a Kleinian group. Then ∂N is a finite union of Riemann surfaces of finite type.*

To prove this theorem, observe that by Theorem 2.5.4, every surface in ∂N is of finite type. Moreover, the sum of the dimensions of the Teichmüller spaces of the components is dominated by the dimension of $H^1(\Gamma; \mathfrak{sl}(2,\mathbb{C}))$ (the inequality in this direction does not need Sullivan's Theorem 2.5.3). The only hyperbolic Riemann surface of finite type with a 0-dimensional Teichmüller space is a thrice-punctured sphere. It follows by a dimension count that there are only finitely many components of ∂N which are not thrice-punctured spheres. Now, a theorem of McCullough ([21]) says that every compact subsurface of ∂N is isotopic into an essential subsurface of $\partial\mathcal{C}$. Since $\mathcal{C}$ is a 3-manifold and a $K(\Gamma, 1)$,

$$\chi(\partial\mathcal{C}) = -2\chi(\mathcal{C}) = -2\chi(\Gamma)$$

so one obtains an *a priori* bound on the number of such thrice-punctured sphere components.

If Γ does not contain parabolic elements, ∂N is compact, and homeomorphic to a union of components of $\partial\mathcal{C}$ (those which cut off geometrically finite ends). The part of N cobounded by corresponding components of ∂N and $\partial\mathcal{C}$ are homeomorphic to products. One way to see this is to take the corresponding components of $\partial\mathcal{C}$ to actually be components of $\partial C(M)$, and then the fibers of the nearest point retraction are intervals which give the end a product structure.

An end of M is *geometrically infinite* if it is not geometrically finite. An end is *tame* if it is homeomorphic to a product $S \times (0, \infty)$ where S is the corresponding component of $\partial\mathcal{C}$. An end which is not tame is said to be *wild*.

Based on his experience with geometrically finite groups, Marden [18] made the following conjecture:

Conjecture 3.2.6 (Marden's Tameness Conjecture [18])**.** *Let Γ be a Kleinian group. Then every end of $\mathbb{H}^3/\Gamma$ is tame. Equivalently, if Γ is torsion free, $M = \mathbb{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold.*

Example 3.2.7 (Doubly degenerate group)*.* Let M be a closed hyperbolic 3-manifold which fibers over S^1 with fiber a surface S. This fibering induces a short exact sequence of fundamental groups

$$0 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 0$$

The hyperbolic structure on M determines an isomorphism from $\pi_1(M)$ to a Kleinian group Γ. Since M is closed, Γ is a uniform lattice, and $\Lambda(\Gamma) = S^2_\infty$. The subgroup $\pi_1(S)$ corresponds to a normal subgroup Γ_S of Γ. Since S is closed, Γ_S is also finitely generated, and is therefore a Kleinian group. However, since the index of $\pi_1(S)$ in $\pi_1(M)$ is infinite, Γ_S is not a lattice. The quotient $\widehat{M} := \mathbb{H}^3/\Gamma_S$ is homeomorphic to the infinite cyclic cover of M corresponding to the surface subgroup; i.e. $\widehat{M}$ is homeomorphic to $S \times \mathbb{R}$, and has two ends. However, since Γ_S is normal in Γ, the limit set $\Lambda(\Gamma_S)$ is a closed, nonempty Γ-invariant subset of S^2_∞, and is therefore equal to all of $\Lambda(\Gamma) = S^2_\infty$; see Figure 2. In particular, the ends of $\widehat{M}$ are geometrically infinite, and tame.

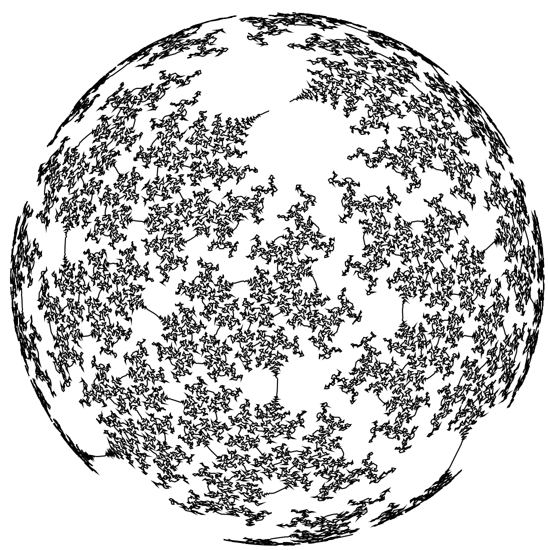

FIGURE 2. The limit set of a doubly degenerate group is all of S^2_∞. As the limit set degenerates, it becomes sphere-filling.

It is not so easy to find an example of a topological 3-manifold with finitely generated fundamental group which is not tame. One silly method is to remove a countably infinite discrete set of points from a noncompact manifold M; however, this gives rise to a manifold whose π_2 is infinitely generated (even as a π_1-module). To give an example of a noncompact wild 3-manifold M with $\pi_1(M)$

finitely generated and $\pi_2(M) = 0$ is much more delicate; a famous example is due to Whitehead [38]:

Example 3.2.8 (Whitehead's manifold). Let $W \subset S^3$ be the Whitehead link. This is an example of a *Brunnian link*: both components are unknots, but each can only be shrunk to a point when the other is removed. Label the components $W = W_1 \cup W_2$ where each W_i is an unknot in S^3. Since each W_i is the unknot, there is a homeomorphism

$$\phi : \overline{S^3 - N(W_1)} \to N(W_2)$$

where the overline denotes closure, and N denotes a regular neighborhood. Since $N(W_2) \subset S^3 - N(W_1)$, this map can be iterated. We denote $S_1 = \overline{S^3 - N(W_1)}$ and inductively define $S_{i+1} = \phi(S_i)$. See Figure 3.

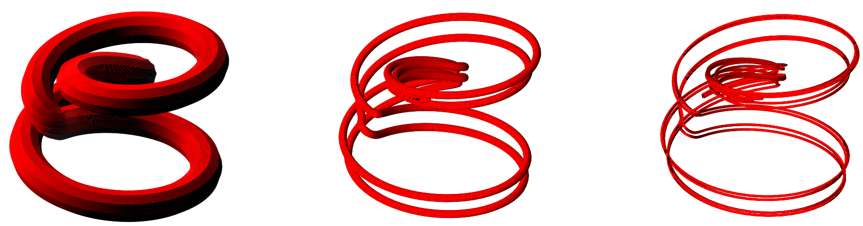

FIGURE 3. Successive approximations to S_∞

Set $S_\infty = \cap_i S_i$ and define $M = S^3 - S_\infty$. M is the Whitehead manifold. Each S_i is a solid torus in S^3 whose core is unknotted. Moreover, a meridian loop α_i representing the generator of $\mathbb{Z} = \pi_1(S^3 - S_i)$ maps to the zero element in $\pi_1(S^3 - S_{i+1})$ under inclusion. Since $\pi_1(M)$ is the direct limit of the groups $\pi_1(S^3 - S_i)$ under inclusion, we see that $\pi_1(M) = 0$. Note that M is noncompact, and has one end, since S_∞ is connected. Notice too that $\pi_2(S^3 - S_i) = 0$ for all i, since $S^3 - S_i$ is homotopy equivalent to a circle, and therefore $\pi_2(M) = 0$.

However, the image of α_i continues to be nontrivial in $\pi_1(S^3 - (S_1 \cup S_j))$ for all j, and therefore any homotopy of α_i to a point in M must intersect S_1. As $i \to \infty$, the loops α_i can be taken to exit any compact subset of M. Since $H_1(M) = 0$, if M is homeomorphic to the interior of a compact manifold, the boundary component of such a compact manifold must be a sphere. This would imply that every end must be homeomorphic to a product $S^2 \times \mathbb{R}$. But any loop in $S^2 \times \mathbb{R}$ can be shrunk to a point in $S^2 \times \mathbb{R}$, contrary to the property of the α_i established above. It follows that M is not tame.

Note that M is an example of an open, contractible 3-manifold which is not homeomorphic to $\mathbb{R}^3$, and is thus a counterexample to a naive generalization of the Poincaré conjecture.

Whitehead's manifold can be turned on its head, and thought of as an increasing *union* of solid tori, where the inclusion of each torus into the next is isotopically knotted, but homotopically trivial.

More generally, wild ends can be produced by taking increasing unions of handlebodies where the image of π_1 of each handlebody eventually stabilizes to some fixed group, but where the images are knotted at each stage. This is the picture to have in mind of a hypothetical "wild end".

4. Lecture 3 — Tameness implies Ahlfors' Measure Conjecture

4.1. **Simply degenerate and homologically degenerate.** The material here is largely based on arguments due to Thurston, found in §8 and §9 of [35]. Thurston's argument does not show that tameness implies Ahlfors directly, but derives this consequence from a more geometric property.

The defining property of a topological product $S \times \mathbb{R}$ is that it is filled up by surfaces $S \times$ point. A *simply degenerate* end is one in which these surfaces can be chosen to have desirable geometric properties.

Definition 4.1.1. A surface S with a path metric is said to be $\text{CAT}(-1)$ if the induced path metric on the universal cover $\widetilde{S}$ is $\text{CAT}(-1)$ as a metric space; i.e. if geodesic triangles are "thinner" than comparison triangles in hyperbolic space.

We say that a map $f : S \to M$ into a Riemannian manifold is $\text{CAT}(-1)$ if the induced path metric on S makes it $\text{CAT}(-1)$. By Gauss-Bonnet, a $\text{CAT}(-1)$ surface has an area which is bounded by its genus. Thus, away from the thin part of M, the image of a $\text{CAT}(-1)$ surface has bounded diameter.

Note that a smooth surface is $\text{CAT}(-1)$ if and only if its Gauss curvature is ≤ -1 pointwise.

Example 4.1.2. If $f : S \to M$ is harmonic, where S is a Riemann surface and M is hyperbolic, then $f(S)$ is $\text{CAT}(-1)$.

Example 4.1.3. If $S \subset M$ is a minimal surface, where M is hyperbolic, then S is $\text{CAT}(-1)$.

Definition 4.1.4. An end $\mathcal{E}$ of a hyperbolic 3-manifold homeomorphic to $S \times (0, \infty)$ is *simply degenerate* if there are a sequence of $\text{CAT}(-1)$ surfaces S_i exiting $\mathcal{E}$ which are homotopic to $S \times$ point in $\mathcal{E}$.

In particular, a simply degenerate end has cross sections of uniformly bounded area, just like an end of a cyclic cover of a manifold fibering over a circle (c.f. Example 3.2.7).

The key geometric property of a simply degenerate end is the existence of the family of homologically essential "barrier" surfaces S_i of bounded area. We introduce the following (non-standard) definition:

Definition 4.1.5. An end $\mathcal{E}$ of a hyperbolic 3-manifold with finitely generated fundamental group is *homologically degenerate* if there are a sequence of $\text{CAT}(-1)$ surfaces S_i of bounded genus exiting $\mathcal{E}$, which homologically separate the Scott core $\mathcal{C}$ from infinity.

Work of Canary [8] and Canary-Minsky [9], building on work of Thurston [35], implies that a homologically degenerate end is simply degenerate. In particular, to prove Marden's Tameness Conjecture, it suffices to show that every end of a hyperbolic 3-manifold is either geometrically finite or homologically degenerate. We will outline an argument which does precisely this in § 5.

Remark 4.1.6. In fact, work of Souto [32] shows that the hypothesis that the surfaces should be $\text{CAT}(-1)$ can be dropped in the definition of homologically degenerate, but this fact is superfluous for these notes.

4.2. **Pleated surfaces.** Historically, the most important class of $\mathrm{CAT}(-1)$ surfaces were *pleated surfaces*, as defined by Thurston.

Definition 4.2.1. Let M be a hyperbolic 3-manifold. A *pleated surface* is a complete hyperbolic surface S of finite area, together with an isometric map $f : S \to M$ which takes cusps to cusps, and such that every $p \in S$ is in the interior of a straight line segment which is mapped by f to a straight line segment.

The set of points $L \subset S$ where the line segment through p is unique is called the *pleating locus*. It turns out that L is a *geodesic lamination*; i.e. a closed union of disjoint geodesics on S, and that f is totally geodesic on the complement of L.

Example 4.2.2. The boundary of the convex core $\partial C(M)$ is a pleated surface.

A homotopy class of loop in a hyperbolic manifold is represented by a unique closed geodesic. So closed leaves in the pleating locus L must map to the unique geodesic in M in their homotopy class. Similarly, every leaf $l \subset L$ whose closure contains only closed leaves has only one possibility for its image in M.

Suppose L is a geodesic lamination on a surface S for which every leaf is either closed, or spirals around closed leaves at either end. Suppose further that complementary regions to L are all ideal triangles (i.e. triangles with their endpoints removed). Suppose $[f] : S \to M$ is a homotopy class of map for which $[f]_*(l)$ is essential in $\pi_1(M)$ for every closed leaf l of L. Then $[f]$ contains a unique representative pleated along a sublamination of L for some choice of hyperbolic structure on S, by first straightening $[f]$ on the closed leaves, then on the spiraling leaves, and finally on the complementary triangular regions (every three geodesics in $\mathbb{H}^3$ with endpoints identified in pairs bound a unique totally geodesic ideal triangle). Note that the pleating locus might be strictly smaller than L, if the image $f(S)$ "accidentally" happens to be totally geodesic along some components of $f(L)$.

The set of geodesic laminations $\mathcal{L}(S)$ on a (closed) hyperbolic surface can be topologized with the geometric topology, where a neighborhood of L consists of all laminations L' which have leaves near every point of L, and nearly parallel to the leaves of L. This topology is not Hausdorff, since if $L \subset L'$ then L' is contained in every neighborhood of L.

As the hyperbolic structure on S varies, the laminations vary, but the topological space $\mathcal{L}(S)$ is unique; that is, a homeomorphism $g : S \to S'$ induces a homeomorphism $\mathcal{L}(S) \to \mathcal{L}(S')$. It therefore makes sense to talk about the space $\mathcal{L}(S)$ when S is merely a topological surface, and ask which laminations in $\mathcal{L}(S)$ contain the pleating locus for a pleated surface in a fixed homotopy class of map $[f] : S \to M$.

If $\mathcal{E}$ is a *tame* end, the product structure $\mathcal{E} = S \times (0, \infty)$ determines a homotopy class of map $[f]$ whose representatives include maps to the slices $S \times$ point. In this case, one looks at all possible limits in $\mathcal{L}(S)$ which arise by realizing sequences L_i in such a way that the corresponding pleated surfaces $f_i(S)$ exit the end $\mathcal{E}$. It turns out that the union of these limits is essentially itself a lamination, called the *ending lamination* $L(\mathcal{E})$ associated to $\mathcal{E}$.

If Γ is parabolic-free, an ending lamination is *filling* in S; that is, complementary regions are all finite sided ideal polygons. If Γ contains parabolics, the ending lamination typically fills some subsurface $R \subset S$. Boundary components of R give rise to accidental parabolics in Γ whose corresponding $\mathbb{Z}$ cusps separate one relative end from another.

There is a technical issue here, which is that one must think of the pleated surfaces which arise as determining *measured* geodesic laminations, and one takes limits in the space $\mathcal{PML}(S)$ of (projectively) measured geodesic laminations, but then forgets the measure. The natural map $\mathcal{PML}(S) \to \mathcal{L}(S)$ obtained by forgetting the measure is continuous, so ending laminations are themselves geometric limits of sequences in $\mathcal{L}(S)$.

If $\mathcal{E}$ is simply degenerate, and S_i is a sequence of pleated surfaces which exits $\mathcal{E}$, the ending lamination is the "limit" of the pleating laminations of the S_i.

One should think of an ending lamination as a "residual" conformal parameter associated to a simply degenerate end, analogous to the conformal structure on a geometrically finite end.

Conjecture 4.2.3 (Thurston's Ending Lamination Conjecture [36])**.** *Let M be a tame hyperbolic* 3*-manifold with finitely generated fundamental group. Then M is determined up to isometry by its topological type, by the conformal geometry on the geometrically finite ends, and by the ending laminations on the geometrically infinite ends.*

This Conjecture is now a Theorem, proved by Minsky [25] and Brock-Canary-Minsky [6]. They show how to build a "model" geometry for a tame end directly from an ending lamination, which is quasi-isometric to any hyperbolic structure on an end with that ending lamination. Since quasi-isometric homeomorphisms between hyperbolic manifolds determine quasiconformal conjugacies between the corresponding Kleinian groups, Theorem 2.5.4 shows that this is enough data to reconstruct the Kleinian group Γ corresponding to $\pi_1(M)$, up to a choice of conformal parameter on Ω/Γ.

4.3. **Harmonic extension.**

Definition 4.3.1. Let $p \in \mathbb{H}^3$. A unit vector $v \in UT_p\mathbb{H}^3$ determines a geodesic ray r_v which starts at p, is tangent at p to v, and limits to a well-defined endpoint $e(v) \in S^2_\infty$. The map $e : UT_p\mathbb{H}^3 \to S^2_\infty$ is called the *visual map*, and is a homeomorphism. The *visual measure* on S^2_∞ *as seen from p* is the push-forward of the angular measure on the unit tangent sphere $UT_p\mathbb{H}^3$ (normalized to have total mass 1) to the sphere at infinity under the visual map.

The visual measures as seen from any two points are in the same measure class, which is also the measure class of the Lebesgue measure on S^2_∞. A random walk on $\mathbb{H}^3$ escapes to infinity with probability 1. By radial symmetry, the visual measure on S^2_∞ as seen from p is the hitting measure for a random walk which starts at p. It follows that if m is a non-negative L^1 function on S^2_∞, the harmonic extension h to $\mathbb{H}^3$ is defined by setting $h(p)$ for each $p \in \mathbb{H}^3$ to be the average of m in the visual measure as seen from p.

We sketch the proof of the following theorem, due to Thurston:

Theorem 4.3.2 (Thurston [35], Corollary 8.12.4)**.** *Let $\Gamma = \pi_1(M)$ where every end of M is either geometrically finite or homologically degenerate. Then either Λ has measure* 0*, or $\Lambda = S^2_\infty$. Moreover, in the latter case, the action of Γ on Λ is ergodic.*

Here an action is said to be *ergodic* if every measurable Γ-invariant set has measure either 0 or 1. Theorem 4.3.2 follows from the following analytic proposition:

Theorem 4.3.3 (Thurston [35], Theorem 8.12.3)**.** *If every end of M is either geometrically finite or homologically degenerate, then for every non-constant positive harmonic function h on the convex core $C(M)$ one has*

$$\inf_{C(M)} h = \inf_{\partial C(M)} h$$

Note in particular that Theorem 4.3.3 implies that if $M = C(M)$ (which happens in Example 3.2.7) then M admits no non-constant bounded harmonic functions at all.

To see that Theorem 4.3.3 implies Theorem 4.3.2, let A be a Γ-invariant measurable subset of Λ (for instance, $A = \Lambda$). Let $\chi_A : S^2_\infty \to \mathbb{R}$ be the characteristic function of A, which is equal to 1 on A and 0 elsewhere. First we suppose $\partial C(M)$ is nonempty.

Let h be the harmonic extension of χ_A to $\mathbb{H}^3$ (i.e. the function on $\mathbb{H}^3$ which at every point p is equal to the average of χ_A in the visual measure as seen from p). Since A is Γ-invariant, both h and $1-h$ descend to positive harmonic functions on M. By the definition of a convex hull, at each point $p \in \partial C(\Lambda)$ the visual measure of Ω is at least $1/2$. It follows that there is an inequality $h \le 1/2$ on $\partial C(M)$. If A has positive measure, let $z \in A$ be a point of density for A. Then there is some Euclidean ball $D \subset S^2_\infty$ centered at z for which the measure of $A \cap D$ is almost equal to the measure of D. Let $q \in \mathbb{H}^3$ be a point for which the visual measure of D as seen from q, and therefore that of A, is close to 1. At such a point, h is close to 1. But this shows that $1-h$ violates Theorem 4.3.3, so A had to have measure 0. If $\partial C(M)$ is empty (equivalently, if $\Lambda = S^2_\infty$), then near points of density of A and $S^2_\infty - A$ one obtains different values for h, again contrary to Theorem 4.3.3; it follows that at least one of them has measure 0.

Sketch of proof of Theorem 4.3.3: We give two proofs, the first using random walks, the second using harmonic functions.

The basic idea of the first proof is that a random walk on $C(M)$, with probability 1, cannot exit a homologically degenerate end $\mathcal{E}$. Consequently, from the point of view of harmonic analysis, $C(M)$ is indistinguishable from a compact manifold, and the infimum of any positive harmonic function must be attained on the boundary, by the maximum principle.

Let $\mathcal{E}$ be a homologically degenerate end, and let S_i be a sequence of CAT(-1) surfaces of bounded genus which homologically separate $\mathcal{C}$ from the end. For each i, let R_i be the region in $\mathcal{E}$ "between" S_i and S_{i+1}. Since R_i is compact, a random walk which enters R_i must leave it eventually. Since the size of the barriers S_i and S_{i+1} are comparable, the random walk is just as likely to leave R_i through S_i as it is through S_{i+1}. If we keep track of which regions R_i the random walk enters, in order, we get a sequence of indices which looks like a random walk on the positive integers. As is well-known, a random walk in 1 dimension is recurrent, so with probability 1, a random walk which enters $\mathcal{E}$ will return to $\mathcal{C}$ eventually. On the other hand, a random walk which exits $C(M)$ has at least a 50% chance of never returning to the convex hull, since for points outside $C(M)$, the domain of discontinuity contains a round open disk with size at least half of the visual sphere. It follows that with probability 1, a random walk on M will eventually exit $C(M)$ and never come back; or, if $M = C(M)$, will be recurrent.

We now give a second proof, using harmonic functions. Let h be a positive harmonic function, and let ϕ_t be the flow generated by the vector field $-\text{grad}\, h$. Since h is harmonic, the flow ϕ_t is volume-preserving. The CAT(-1) surfaces S_i act like "barriers", which prevent any definite amount of volume flowing by ϕ_t out of $\mathcal{E}$. Consequently, almost every flowline of ϕ_t must hit $\partial C(M)$, proving the Theorem.

We give more details. If we integrate the velocity $\|\text{grad}\, h\|$ of ϕ_t along a flowline, we obtain an estimate

$$\int_x^{\phi_T(x)} \|\text{grad}\, h\|\, ds = h(x) - h(\phi_T(x)) \le h(x)$$

assuming $T > 0$, since h is positive. Note that restriction to a subset can only decrease the integral of the positive function $\|\text{grad}\, h\|$, so the inequality remains true whenever we integrate over any (finite) subset A of a flowline.

If A is a (possibly disconnected) subset of a flowline of finite length $\ell(A)$, and $T(A)$ is the total time that the flowline spends in A, then

$$T(A) = \int_A \frac{1}{\|\text{grad}\, h\|}\, ds$$

By Cauchy-Schwartz

$$\int \frac{1}{\|\text{grad}\, h\|}\, ds \cdot \int \|\text{grad}\, h\|\, ds \ge \left(\int 1\, ds\right)^2 = \ell(A)^2$$

and therefore

$$T(A) \ge \frac{\ell(A)^2}{\int_A \|\text{grad}\, h\|\, ds} \ge \frac{\ell(A)^2}{h(x)} \tag{1}$$

where x is the initial point of the flow on A.

On the other hand, let S_i be the sequence of CAT(-1) surfaces guaranteed by the definition of homologically degenerate. Since the diameters of the S_i are bounded away from the thin part, we take an infinite subsequence for which the neighborhoods $N(S_i)$ of radius 1 are disjoint (technically, we need to modify the S_i near $\mathbb{Z}$-cusps to guarantee this, but if Γ is parabolic free, we do not need to modify the S_i).

Let $p \in C(M)$ be a point where grad h does not vanish. Then for some sufficiently small ball B about p, the time 1 flow $\phi_1(B)$ pushes B off itself. Because ϕ_t is a gradient flow, for a suitable choice of B, the images $\phi_i(B)$ are all disjoint, where i ranges over the non-negative integers. Since h is harmonic, grad h is divergence-free (i.e. volume preserving), so almost every flowline through B eventually leaves every compact subset of M. There are two possibilities: a flowline either escapes out an end, or hits $\partial C(M)$. If a flowline does not hit $\partial C(M)$, then it must cross every S_i essentially, and therefore intersects each $N(S_i)$ in a curve of total length at least 2. In particular, by the estimate (1), the total time that a flowline spends in $\bigcup_{i=1}^n N(S_i)$ is at least $O(n^2)$.

On the other hand, the volume of $\bigcup_{i=1}^n N(S_i)$ is at most $O(n)$, so since ϕ_t is volume preserving, the measure of the set of flowlines which passes through every S_i is 0. $\square$

5. Lecture 4 — Shrinkwrapping, and the proof of Tameness

We have seen that if Γ is a Kleinian group for which every end of $M = \mathbb{H}^3/\Gamma$ is either geometrically finite or homologically degenerate, then Γ satisfies Ahlfors' Measure Conjecture.

As remarked at the end of § 4.1, a homologically degenerate end in a hyperbolic manifold is simply degenerate, and therefore necessarily tame; the argument uses some basic compactness properties of families of pleated surfaces, and some standard 3-manifold topology. Conversely, Canary [8] showed that every tame end of a hyperbolic 3-manifold with finitely generated fundamental group is either geometrically finite or simply degenerate; thus Canary's theorem together with Theorem 4.3.2 reduces Ahlfors' Measure Conjecture to Marden's Tameness Conjecture.

It turns out that one can directly prove that every geometrically infinite end is homologically degenerate, thereby proving Ahlfors' Conjecture without technically appealing to Canary's theorem. However, Canary's work greatly clarifies the relationship between tameness and homological degeneracy, and his point of view has shaped and continues to shape the modern theory of Kleinian groups.

The arguments in this section are based largely on [7]. It should be stressed that Ian Agol [1] gave an independent proof (based on quite different arguments) of Marden's Tameness Conjecture, and therefore of Ahlfors' Measure Conjecture, at about the same time.

5.1. **Exiting sequence.** Let Γ be a Kleinian group, with $M = \mathbb{H}^3/\Gamma$, and let $\mathcal{E}$ be a geometrically infinite end. At least in the parabolic-free case, $\mathcal{E}$ is contained in the compact core $C(M)$. Now, the compact core is the closure of the union of the set of closed geodesics in M, so there are certainly closed geodesics which intersect every neighborhood of infinity in $\mathcal{E}$. Bonahon [5] strengthened this fact considerably:

Theorem 5.1.1 (Bonahon [5])**.** *Let M be a hyperbolic 3-manifold with finitely generated fundamental group. An end $\mathcal{E}$ of the thick part of M is geometrically infinite if and only if every neighborhood of infinity in $\mathcal{E}$ contains a closed geodesic.*

In particular, geometrically infinite ends can be characterized by the property that they contain an exiting sequence of geodesics γ_i which leaves every compact subset of M.

Given M, a Scott core $\mathcal{C}(M)$ is a union of 1-handles (possibly empty) attached to a compact 3-manifold X with incompressible boundary. If we split M along all the boundary components of X, and let M' be the component which contains $\mathcal{E}$, then we see that $\pi_1(M')$ is a free product of finitely many finitely generated free groups, and closed orientable surface groups. We define M' to be the *end manifold* of M. The inclusion $M' \to M$ induces a monomorphism in π_1, and M' is homotopy equivalent to the cover $\widehat{M}$ of M which corresponds to the subgroup $\pi_1(M')$.

Remark 5.1.2. In the case that $\pi_1(M')$ is a closed surface group (i.e. there are no 1-handles in M') Bonahon [5] showed that $\mathcal{E}$ is tame. In this case, $\partial M' = S$ is incompressible in M. The exiting sequence γ_i determine an infinite sequence of conjugacy classes in $\pi_1(S)$. Bonahon shows that in this case, one can choose the geodesics γ_i in such a way that they represent *simple* loops a_i on S. Let L_i be a geodesic lamination on S with triangle complementary regions, containing a_i as a leaf and such that every other leaf spirals around a_i in both directions. Then some subset of L_i can be realized as the pleating lamination of some pleated surface $f_i : S \to M$. By Gauss-Bonnet, the diameter of $f_i(S)$ (in the thick part) is uniformly bounded in terms of Euler characteristic. Since $f_i(S)$ contains γ_i, and since γ_i go to infinity in $\mathcal{E}$, it follows that the $f_i(S)$ go to infinity in $\mathcal{E}$. This implies that $\mathcal{E}$ is simply degenerate, and tameness follows.

The completely opposite case, that $\pi_1(M')$ is a finitely generated free group, is perhaps the "typical" remaining case to understand. For simplicity therefore, we concentrate in the sequel on the case that $\mathcal{C}(M)$ is a handlebody of genus g.

5.2. **Engulfing and 2-incompressibility.** Our setup is that we have a hyperbolic manifold M with $\pi_1(M) = F_g$, the free group on g generators, and M has a single geometrically infinite end $\mathcal{E}$. By Bonahon's Theorem 5.1.1, there is a sequence γ_i of closed geodesics which exit $\mathcal{E}$. We let W_i be a nested sequence of compact submanifolds which exhaust M, with $\gamma_i \subset W_i$. We can take the W_i to be handlebodies, though typically of genus $g_i > g$, and typically with each W_i knotted in a complicated way in W_{i+1}.

Since $\pi_1(M)$ is free, the surfaces ∂W_i are never incompressible in W_i, but by enlarging them if necessary, we can assume that each ∂W_i is incompressible in $M - W_i$.

Define

$$\Delta_i = \bigcup_{j=1}^{i} \gamma_j \subset W_i$$

We say a properly embedded disk $D \subset W_i$ is a 2-*compressing disk* if ∂D is essential in W_i, and if D intersects Δ_i transversely at only one point. If a 2-compressing disk exists, we say ∂W_i is 2-*compressible*; otherwise it is 2-*incompressible*.

By a purely topological argument, using only the fact that each γ_i is essential in $\pi_1(M)$, one shows that after possibly passing to a (still infinite) subsequence of the γ_i, one can assume that each ∂W_i is 2-incompressible. This is the *Infinite End-Engulfing Theorem*, which is Theorem 5.21 from [7].

5.3. **Shrinkwrapping.** We now show how to use the exhaustion W_i and the geodesics γ_i to produce the desired sequence of CAT(-1) surfaces which exit the end of $\mathcal{E}$, thereby certifying it as simply degenerate.

The basic idea is to use the geodesics Δ_i as obstacles, and to try to produce surfaces (in certain homotopy classes in $W_i - \Delta_i$) which are minimal, subject to the constraint that they do not cross Δ_i. We call the operation of minimizing a surface subject to such a constraint *shrinkwrapping*. A minimal surface in a hyperbolic 3-manifold is CAT(-1); if one can show that shrinkwrapped surfaces have no "distributional curvature" along Δ_i, then they will be CAT(-1).

The local version of this problem is known as the *thin obstacle problem*. Given a domain $U \subset \mathbb{R}^2$ and an obstacle $X \subset U$, $\psi : X \to \mathbb{R}$, the problem is to find $h : U \to \mathbb{R}$ with $h|_{\partial U}$ prescribed, so that $h|_X \geq \psi$, while minimizing some functional

$$\mathcal{F}(h) := \int_U F(x, h, \nabla h) dx$$

Typically, we want to minimize $\mathcal{F}(h)$ as h varies in some Sobolev space of functions, and the integral kernel F should be "elliptic" in a suitable sense. Examples are $F = |\nabla h|^2$ (minimize energy) and $F = (1 + |\nabla h|^2)^{1/2}$ (minimize area of the graph).

This problem is well-studied, and some useful references are [28],[12] and [16].

The technical theorem we prove is the following, which is Theorem 0.8 from [7]:

Theorem 5.3.1 (Calegari-Gabai, Shrinkwrap Theorem [7])**.** *Let M be a complete, orientable, parabolic free hyperbolic 3-manifold, and let Γ be a finite collection of pairwise disjoint simple closed geodesics in M. Furthermore, let $S \subset M - \Gamma$ be a closed embedded 2-incompressible surface which is either nonseparating in M, or separates some component of Γ from another. Then S is homotopic to a* $\mathrm{CAT}(-1)$ *surface T via a homotopy which is embedded and disjoint from Γ up to the last instant. Furthermore, if T' is any other surface with this property,* $\mathrm{area}(T') \geq \mathrm{area}(T)$.

The proof of this theorem uses existence results for embedded minimal surfaces due to Meeks-Simon-Yau [23]. If M is a closed, irreducible Riemannian 3-manifold and S is an incompressible surface in M, then either S bounds a twisted I-bundle, or else S is isotopic to a locally least area minimal surface. If M is noncompact, one needs to impose some topological conditions on S to ensure the existence of a minimal representative. If M has boundary, one further needs to impose the condition that the boundary is *mean convex*; i.e. that the mean curvature vector does not point out of the boundary at any point. Note that a minimal surface is mean convex from either side.

We deform the metric in a neighborhood $N(\Gamma)$ of Γ, adding a radially symmetric "bubble" of positive curvature on each cross-section, so that the boundary of the neighborhood becomes totally geodesic. Then $M - N(\Gamma)$ has mean convex boundary, and one can find a minimal representative. As N shrinks down to Γ, the surfaces so obtained have a convergent subsequence which is in a fixed isotopy class, and converges uniformly to the shrinkwrapped surface T. Away from Γ, T is a minimal surface. Hence it is C^ω in $M - \Gamma$, and has curvature bounded above by -1 there. Along Γ, T is C^ω from each side at interior points of $\Gamma \cap T$, and is $C^{1+1/2}$ at frontier points. A degree argument shows that the frontier points contain no distributional curvature, and therefore T is $\mathrm{CAT}(-1)$.

5.4. **PL-wrapping.** Soma [31] adapted the shrinkwrapping argument to the piecewise totally geodesic context. The inclusion $S \to M - \Gamma$ is injective in π_1, so it determines a covering space $\widehat{M}$. The path metric on M lifts to a path metric on $\widehat{M}$. By 2-incompressibility, the metric completion is a $\mathrm{CAT}(-1)$ metric space, which is locally isometric to hyperbolic space away from the boundary. Essential loops in S have geodesic representatives in $\widehat{M}$, so one can build a $\mathrm{CAT}(-1)$ representative of S by pleating (compare with the construction described in § 4.2).

Soma's construction has fewer analytic details than [7], and may be used in its place to give a somewhat simplified proof of Tameness.

5.5. **Logic of the proof.** The basic argument employed in [7] to produce a sequence of $\mathrm{CAT}(-1)$ surfaces is known as *double shrinkwrapping*. Naively, one would like to shrinkwrap the surfaces ∂W_i to obtain an exiting sequence of $\mathrm{CAT}(-1)$ surfaces. However, *a priori*, we have no control on the genus of the surfaces ∂W_i, and therefore no control on their diameter, even after shrinkwrapping. So we must employ a suitable covering trick to find surfaces of the correct genus.

Given W_i as guaranteed by the Infinite End-Engulfing Theorem, we shrinkwrap ∂W_i rel. Δ_i to produce a new (possibly degenerate) "submanifold" W_i' isotopic in M to W_i. We let G_i be the subgroup of $\pi_1(W_i)$ corresponding to the inclusion of the core $\mathcal{C}(M)$ into W_i, and let X_i be the covering space of W_i corresponding to G_i. Let X_i' be the corresponding covering space for W_i'. Since every geodesic in Δ_i is homotopic into $\mathcal{C}(M)$, the collection Δ_i lifts to $\widetilde{\Delta}_i$ in X_i'. Now, W_i and W_i' are handlebodies of genus g_i, whereas X_i and X_i' have fundamental group isomorphic to F_g. The manifold X_i' has a natural compactification Y_i making it homeomorphic to a closed handlebody of genus g; we "push" the surface ∂Y_i into X_i' to obtain a surface of the *correct* genus g. We now shrinkwrap ∂Y_i in X_i' relative to Δ_i, where the fact that $\partial X_i'$ was minimal lets us use it as a barrier surface. Of course, ∂Y_i might not be 2-incompressible rel. $\widetilde{\Delta}_i$; but since the genus is fixed independently of i, after performing only finitely many compressions, we can make it 2-incompressible rel. all but at most g of the components of $\widetilde{\Delta}_i$.

The result is a $\mathrm{CAT}(-1)$ surface $\widetilde{T}_i$ of genus g in X_i'. Projecting to M gives us a $\mathrm{CAT}(-1)$ surface T_i of genus g in M. After relabeling if necessary, we see that each T_i is homologically essential, and separates γ_i from the end of $\mathcal{E}$. By Gauss-Bonnet, each T_i has bounded area, and therefore bounded diameter in the thick part of M, so the T_i must exit the end. Since the genus of each T_i is equal to the genus of the corresponding component of $\partial\mathcal{C}(M)$, $\mathcal{E}$ is homologically degenerate.

The cases when $\pi_1(M)$ contains surface group summands, or when Γ contains parabolic elements, are similar.

This proves Marden's Conjecture and Ahlfors' Measure Conjecture.

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