

Summary of papers

Published papers

- (1) *Peripheral separability and cusps of arithmetic hyperbolic orbifolds*. *Algebr. and Geom. Topol.* **4** (2004), 721-755.

The main result of this article is the topological (and smooth) classification of cusp cross sections of arithmetic real, complex, and quaternionic hyperbolic n -orbifolds. The main tools are the subgroup separability of virtually solvable subgroups of arithmetic lattices and classification of algebraic injections of peripheral groups into the isometry groups of these symmetric spaces. From these results, I also showed every Nil 3-manifold arises (smoothly) as a cusp cross section of a complex hyperbolic 2-orbifold and construct examples of manifolds modeled on $(2n - 1)$ -dimensional Heisenberg groups that cannot be topologically realized as cusp cross sections of any arithmetic complex hyperbolic n -orbifold for all $n > 2$. Examples modeled on the quaternionic analog of the Heisenberg groups also exist that cannot topologically arise as cusp cross sections of quaternionic hyperbolic n -manifolds for all $n > 1$; for $n = 1$, these manifolds are real hyperbolic 4-manifolds.

The main tools used are algebraic group theory, algebraic number theory, the theory of bilinear and hermitian forms over global fields, almost crystallographic groups, nilpotent Lie algebras and groups, Tits' classification of algebraic groups, and Galois cohomology (see my thesis for the portions where Galois cohomology arise).

- (2) *Finite subgroups of arithmetic lattices in $U(2, 1)$* , *Geom. Dedicata.* **122** (2006) 135–144.

In this article, I investigated finite subgroups of arithmetic lattices in $U(2, 1)$ and more generally $U(n, 1)$. Let p be a prime. If Γ is arithmetic lattice in $U(p, 1)$ arising from a cyclic division algebra and $Q < \Gamma$ is a finite subgroup, then Q is cyclic; I call this result the Cyclic Subgroup Theorem. In contrast, I prove that every finite group can be embedded in an arithmetic lattice in $U(n, 1)$ arising from hermitian forms and for some sufficiently large n . These results can be viewed as refined versions of results first established by Jordan in 1868 who investigated finite subgroups on $GL(n, \mathbf{Z})$. It is worth noting that the primality condition on p can be dropped though the proof is somewhat more involved and has never appeared in print. Indeed, I have worked out which families of arithmetic constructions have obstructions preventing the embedding of a finite group and might write a second paper on this topic; it seems better suited for a graduate student to work out.

The main tools I use are work of Amitsur, representation theory, theory of hermitian forms over global fields, and Tits' classification of arithmetic groups.

- (3) (w/ C. J. Leininger) *Separable subgroups of mapping class groups*, *Topology Appl.* **154** (2007), 1–10.

In this article, we investigate separable subgroups of mapping class groups and $\text{Out}(F_n)$. Equivalently, separability is a measure of the richness of the profinite topology or the structure of the profinite completion of these groups; the structure of the profinite completion is very much at the heart of the congruence subgroup problem for these groups (see [11] and [13]). Using [1], we prove virtually solvable subgroups are separable. We show that stabilizers and trivializers of subvarieties of a variety with an algebraic group action (birational transformations) are separable; these results require the variety, subvariety, and the action to all be defined over number fields. Mapping class groups and $\text{Out}(F_n)$ have many natural actions on varieties; for example, the \mathbf{G} -character varieties of a surface or free group for a k -algebraic

reductive group \mathbf{G} has a natural mapping class group or outer action. The resulting stabilizers/trivializers of subvarieties under these actions, though generically trivial, provide interesting subgroups of mapping class groups that mimic fundamental groups of totally geodesic submanifolds. To my knowledge, this article is the first to study this class of subgroups and many interesting and well known subgroups arise as stabilizers and trivializers of subvarieties.

We employ low dimensional topology, structure theory for mapping classes, algebraic geometry, algebraic number theory, and elementary topology and group theory.

- (4) *Arithmetic cusp shapes are dense*, *Geom. Dedicata* **129** (2007), 47–55.

This article is the sequel to [1]. The main result is the classification of similarity classes of flat structures for cusp cross sections of arithmetic real hyperbolic n -orbifolds. This is a considerably more subtle problem than the smooth classification problem. As a corollary, I resolved the orbifold version of Nimershiem's Conjecture of flat structures in cusp cross sections of finite volume hyperbolic n -manifolds. Namely, that the similarity classes of flat structures that can be realized in cusp cross sections of real hyperbolic n -orbifolds is dense. This result is shown by proving two things. First, I classify those flat classes arising from arithmetic orbifolds. This utilizes [1] and Tits' classification of arithmetic constructions. Second, I show that the set of such classes can be viewed as the rational points of a variety and show these points are dense. These results also extend to almost flat manifolds modeled on the Heisenberg groups and their quaternionic analogs.

The main tools I use are affine and Euclidean groups for nilpotent groups, the theory of bilinear and hermitian forms over global fields, algebraic geometry and number theory, and algebraic actions on varieties (this is central to proving the density).

- (5) (w/ C. J. Leininger, W. D. Neumann, and A. W. Reid) *Length and eigenvalue equivalence*, *Int. Math. Res. Not.* **24** (2007), 24 pp.

In this article, we investigate what multiplicities see in the eigenvalue and geodesic length spectra. For symmetric spaces of non-compact type, we produce large families of pairs of finite covers of a fixed manifold that have identical eigenvalue or length sets but have unbounded volume ratio. This result shows that multiplicities encode volume, a known heat invariant of the full eigenvalue spectrum. For negative curved symmetric spaces, we also construct examples that have the same primitive length set and large volume ratio. In addition, we provide examples with identical length sets but different primitive length sets, something that can never happen with multiplicities. The main method is a generalization of Sunada's method.

We utilize results from various areas though the arguments are elementary; classic Lie theory, finite group theory, algebraic and analytic number theory, structure theory of p -adic Lie groups, the Strong Approximation Theorem for algebraic groups, and arithmetic constructions are amongst the topics and tools we employed.

- (6) *Arithmetic lattices and weak spectral geometry*, *Topology, complex analysis, and arithmetic of hyperbolic spaces*, *RIMS* **1571** (2007), 59–81.

In this article, I discuss the use of arithmetic and non-arithmetic manifolds in spectral geometry and geometric problems; producing isospectral, non-isometric manifolds and the inverse Galois problem for isometry groups are two examples covered. I also review the classification of (non-exceptional) arithmetic lattices in the real rank one simple Lie groups; the quasi-split form of F_4 and triality constructions for $SO(8)$ are not covered. These are extended notes from a lecture series I delivered at Kyoto University in December 2006.

- (7) *Cusps of Hilbert modular varieties*, Math. Proc. Cambridge Philos. Soc. **144** (2008), no. 3, 749–759.

In this article, I give the topologically (smooth) classification of cusp cross sections of Hilbert and Hilbert–Blumenthal modular varieties. This classification answered a question asked by Hirzebruch in 1973. I prove that every Sol 3–manifold topologically arises as a cusp cross section of a Hilbert modular surface. This uses the subgroup separability results from [1], analysis of the monodromy representation for torus bundles of tori, algebraic number theory, solvable algebraic group results of Mostow, representation theory, and work of Scott (Thurston) on the possible geometries of 3–manifolds.

I also show that there exist Sol 3–manifolds that cannot (topologically) arise as the cusp cross section of a 1–cusped non-singular Hilbert modular surface. The proof of this last result also uses the Atiyah–Donnelly–Singer Index Theorem, the work of Cheeger–Gromov on the geometric independence of adiabatic limits on certain topological manifolds, the Hirzebruch Signature Theorem, and special values of Hecke L –functions.

- (8) (with L. Louder) *Graphs of subgroups of free groups*, Algebr. and Geom. Topol. **9** (2009), 327–335.

The main result of this article is the construction of an efficient topological model (graph) representing a finitely generated subgroup of free groups that minimizes the Scott complexity. Using this construction, we give a very short proof of a theorem of Dicks that equates the strengthened Hanna Neumann conjecture with the Amalgamated Graph Conjecture. We also answer a question of Culler–Shalen (independently done by R. Kent) on the possible rank of the intersection of a pair of rank two free groups; this question evidently was first posed by Burns. Culler and Shalen have recently used this result to produce new lower bounds on volumes of hyperbolic 3–manifolds. Special cases of the Hanna Neumann Conjecture can be obtained from this work as well. Our tools are topological and rest on earlier ideas of Louder in his important work on Krull dimension.

- (9) *Controlling manifold covers of orbifolds*, Math. Res. Lett. **16** (2009), 651–662.

This article is the final article on the topological and geometric classification of cusp cross sections of arithmetic hyperbolic n –manifolds started in [1] and continued in [4]. The main tool is a generalization of Selberg’s Lemma on the existence of torsion free finite index subgroups in finitely generated linear groups. Namely, for an arithmetic group, I show that every torsion free virtually unipotent group is contained in a torsion free finite index subgroup; the most general result avoids any finite collection of conjugacy classes of semisimple elements. Using this result and [4], I resolve a strong form of Nimershiem’s Conjecture on density of flat similarity structures that occur in cusp cross sections of arithmetic hyperbolic n –manifolds. I also answer a question of Long–Reid and a question from [1] on realizing flat and almost flat manifolds as cusps of arithmetic real, complex, and quaternionic hyperbolic manifolds.

The methods used in the proof of the Selberg Theorem are elementary (Jordan Normal Form, Heine–Borel Theorem, and p –adic analysis) modulo a finiteness lemma that requires classical results on torsion-free, finitely generated, nilpotent groups (Mal’cev Rigidity, nilpotent Lie algebras, log and exp maps, and an algebraic version of Jordan Form established by Borel).

- (10) (w/ K. Bou-Rabee) *Bertrand’s postulate and subgroup growth*, J. of Algebra **324** (2010) 793–819.

Divisibility functions on finitely generated groups measure how difficult it is to establish the non-triviality of a non-trivial element in a group via finite index subgroups. These functions were first introduced by Bou-Rabee in his article on quantifying residual finiteness. Bou-Rabee studied the extremal behavior (or L^∞ –norm on metric balls) of these functions. In this article, we study the average behavior or L^1 –norm of these functions. The main tools needed in proving these functions are L^1 are a

pair of results that can be viewed as generalizations of Bertrand's postulate to finitely generated linear groups. The proof of the strong form of Bertrand's postulate uses the Lubotzky Alternative and the Strong Approximation Theorem. Using this result, we prove the L^1 -norm of the divisibility function studied in [13] is finite. The weak form of Bertrand's postulate for groups avoids the Strong Approximation Theorem (a result that requires the classification of finite simple groups) and works uniformly (i.e., no separate cases based on the structure of the group). It should be noted that there are many difficulties that arise from not using the Strong Approximation Theorem! Using the weak form together with some additional estimates, we prove the finiteness of the L^1 -norm of the normal divisibility function; this implies the finiteness of the L^1 -norm of the divisibility function. Indeed, the strong form of Bertrand's postulate for groups cannot be used as it cannot ensure the finite index subgroups are normal, an indispensable feature for investigate averages of normal divisibility functions.

We use results from profinite group theory, integration theory on topological groups, algebraic and analytic number theory, p -adic Lie groups, elementary group theory, infinite series, and combinatorics. Indeed, results like the Ratio Test and L'Hopital's rule are quite useful in this article!

- (11) (w/ J. Ellenberg) *Every curve is a Teichmüller curve*, to appear in *Duke Math. J.* 16 pages.

In this article, we prove a pair of results. First, we show that every algebraic curve X defined over $\overline{\mathbf{Q}}$ is bi-rational to a Teichmüller curve. This establishes a converse to a theorem of McMullen who showed Teichmüller curves are always defined over $\overline{\mathbf{Q}}$. The first result is a corollary of our main result. We show every finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$ contained in the level two congruence subgroup and also containing the center, can be realized as a Veech group. This establishes a strong form of the congruence subgroup property for $\mathrm{Mod}(S_{1,1})$, a result first proved by Asada. The proof has two main steps. A proposition that controls normalizers of subgroups in free groups and a simple variational argument that shows that the families of candidate subgroups we produce can generically be used to realize a fixed subgroup as a Veech group.

One reason our work on Veech groups is important is that these groups are difficult to construct and seem to be rather exceptional. Our work not only shows that a rather large class of groups are Veech groups but might hint at a more general result. Namely, that Veech groups are closed under taking finite index subgroups. The primary obstruction to proving results along that line is symmetry and ensure one can break algebraic and geometric symmetry in covers.

This work has recently been used by Avila–Matheus–Yoccoz to show the existence of Teichmüller curves with complementary series; these representations correspond to small eigenvalues in the Laplace spectrum (see [18] below for a bit more on complementary series representations, rate of mixing, and critical exponents).

The main tools we employ are elementary group theory and elementary topology modulo a proposition on normalizers of subgroups in free groups; this proposition gave rise to [19]. The proof of this proposition uses algebraic actions on varieties, algebraic geometry, representation theory, finite simple groups, and the Cebotarev Density Theorem.

Preprints

- (12) *Isospectral locally symmetric manifolds*, submitted. 21 pages.

In 1989, Spatzier produced the first examples of higher rank locally symmetric manifolds that were isospectral and nonisometric. In this article, I extend and strengthen his results. I produce arbitrarily large families of closed, isospectral, non-isometric locally symmetric manifolds when the symmetric space is irreducible. I also produce infinite towers of closed locally symmetric manifolds that are isospectral and

non-isometric. In most settings, arbitrarily large families were not known to exist. In some settings, no pairs were known to even exist; for example, complex hyperbolic 2-space. The previously known tower examples were known only for the $SL(n)$ family.

I also analyze the growth of the maximum number of pairwise isospectral and non-isometric manifolds with volume at most t modeled on a fixed symmetric space X ; this extends and strengthens work of Brooks–Gornet–Gustafson who investigated this function when X is the hyperbolic plane. I prove that this function is super polynomial in general and when X is real hyperbolic n -space, this function has at least exponential growth.

The main idea for the constructions is to produce large families of isospectral manifolds that should be pairwise isospectral. This task is done using Sunada's method and finite groups used by Brooks–Gornet–Gustafson for Riemann surfaces. The main difficulty is ensuring the covers produced are non-isometric. In the arithmetic setting, this control is achieved with a recent result of Belolipetsky and Lubotzky obtained via Bruhat–Tits theory.

Some of the basic tools are algebraic and analytic number theory, algebraic group theory, Tits' classification of arithmetic lattices, p -adic Lie groups, finite simple groups, theory of bilinear and hermitian forms, almost inner automorphisms on nilpotent groups, and delicate counting arguments.

- (13) *Congruence subgroup problem for braid groups: Thurston's proof*, submitted. 17 pages.

In this article, I give a proof of the congruence subgroup property for pure braid groups based on an idea of W. Thurston. This is the fourth proof of this theorem but has the advantage of being quite elementary. In addition, the methods provide elementary proofs of certain results on profinite groups and profinite versions of connecting homomorphisms related to the Birman exact sequence. I also provide a contrast and comparison of some of the previous proofs.

- (14) (w/ A. W. Reid) *The genus spectrum of a hyperbolic 3-manifold*, submitted. 18 pages.

The purpose of this article is to initiate the study of higher dimensional totally geodesic submanifolds of a Riemannian manifold analogous to the geodesic length spectrum. For simplicity, we confine ourselves to hyperbolic 3-manifolds and the study of their totally geodesic surfaces. In this setting, we prove a pair of results. First is a commensurability rigidity theorem. Namely, for arithmetic hyperbolic 3-manifolds, the set of maximal totally geodesic surfaces either determines the commensurability classes of the manifold or is empty. Second is the failure of geometric rigidity. Namely, for any finite volume arithmetic hyperbolic 3-manifold, we show that there are many pairs of finite covers with identical surfaces counted with or without multiplicity. Without multiplicity, like with [5], we show the volume ratio can be arbitrarily large.

The proofs of these results use quaternion algebras, local-to-global principles, algebraic and analytic number theory, finite simple groups, low dimensional hyperbolic geometry (more accurately the geometry locally symmetric manifolds), and measure theoretic results on subsets of primes.

- (15) (w/ K. Bou-Rabee) *Asymptotic growth and least common multiples in groups*, submitted. 12 pages.

In this article, we continue the study of divisibility functions on groups started by Bou-Rabee and studied in [10]. We connect word and subgroup growth via the normal divisibility function. Using a generalization of least common multiples to finitely generated groups, we establish lower bounds (as a function of n) on the L^∞ -norm of the normal divisibility function on the n -ball. Using a somewhat intricate inductive construction of covers, we also show that the divisibility function cannot have a $C \log(Cn)$ lower bound on the n -ball for any constant C . This answers Problem 15.35 in the Kourovka notebook asked by Oleg Bogopolski. We also introduce a function called the residual girth function that we relate to

divisibility functions and subgroup growth. Recently, Kassabov–Mattucci improved one of the lower bounds in this article. Specifically, they improved a power from $1/3$ to $1/2$. More recently, a new proof of our resolution of Problem 15.35 was pointed out to us by a referee. It avoids the use of topology and uses some results on the growth rate of the Landau function; this function on the positive integers n takes the value $L(n)$, where $L(n)$ is the maximum order of an element in the symmetric group $\text{Sym}(n)$.

- (16) (w/ K. Bou-Rabee) *Quantifying Mal'cev Theorem*, working draft (6 pages).

This is the final article on the extremal behavior of divisibility functions of finitely generated linear groups. We give a complete characterization of logarithmic growth rate of the L^∞ -norm of the normal divisibility functions introduced by Bou-Rabee for finitely generated groups in $\text{GL}(n, K)$, where K is an infinite field. This result can be viewed as a quantitative version of the residual finiteness of such groups. Our work establishes that the log-growth rate of the normal divisibility function is either $\log \log n$ or $\log n$. Moreover, $\log \log n$ growth rate occurs if and only if the group is virtually nilpotent; this final statement requires Gromov's theorem on word growth for nilpotent groups.

The methods, as in our previous work [10] [14], are elementary and algebraic; elementary ring and field theory, the Chebotarev Density Theorem, and elementary group theory.

- (17) (w/ A. Mohammadi) *Geometric spectra and commensurability*, working draft (8 pages).

This article continues the studying of totally geodesic submanifolds initiated in [13]. Again, the basic question is how much global geometry is determined by the geometry of totally geodesic submanifolds. In this article, we investigate when the commensurability class can be determined from the totally geodesic submanifolds. The general theme is that when there are interesting collections of submanifolds, the commensurability class is determined by the collection. In contrast, we construct manifolds modeled on products of hyperbolic planes that have the same commensurability classes of totally geodesic submanifolds of dimension greater than one but are not commensurable. However, if one further insists on length relations, the manifolds are forced to then be commensurable. These examples illustrate the delicate nature of these questions and the importance of geometric information in many dimensions.

The methods employed are largely algebraic and rely on classical results in number theory and division algebras; global class field theory, local-to-global principles, ζ -functions for number fields, work of Prasad–Rapinchuk, the geometry of local symmetric manifolds, and Tits' classification of arithmetic constructions. This article is a companion article to the forthcoming [21].

- (18) (w/ B. Farb) *Cohomological dimension spectrum of a group*, working draft (12 pages).

In this article, we investigate the following basic question: Given a finitely presented group Γ , what are the possible cohomological dimensions of the finitely generated subgroups Δ of Γ . Among the real rank one simple Lie groups, lattices in $\text{Sp}(n, 1)$ are considerably more rigid than those in $\text{SO}(n, 1)$ and $\text{SU}(n, 1)$. Our main theorem shows yet another contrast in behavior for quaternion hyperbolic lattices. Namely, we demonstrate that for a finitely generated subgroup Δ of a cocompact lattice Γ of $\text{Sp}(n, 1)$, either Δ is finite index or the cohomological dimension of Δ is strictly smaller than $4n - 1$. An identical result also holds for cocompact lattices in the exceptional isometry group of the Cayley hyperbolic plane. We prove this result in two main steps. Following Misha Kapovich, we relate homological dimension of finitely generated groups with the critical exponent of the group; the critical exponent has both dynamical interpretation via Patterson–Sullivan measures and spectral interpretation via harmonic measures. The main tool required for this step are the natural maps introduced by Besson–Courtois–Gallot and estimates on the Jacobians of these maps. Indeed, a bulk of the work is in generalizing some of the estimates of Besson–Courtois–Gallot. With these maps and our estimates on the Jacobian,

the relationship between the critical exponent and homological dimension is achieved via a topological construction taken from Kapovich and homological triviality results from geometric measure theory. With this relationship, the final step is work of Corlette on critical exponents for discrete subgroups of $\mathrm{Sp}(n, 1)$; this work further ties the critical exponent to representation theory and the exclusion of complementary series representations which give rise to small eigenvalues or faster rates of mixing for the geodesic flow.

- (19) (w/ J. Ellenberg) *Group actions on varieties and isometry groups of hyperbolic manifolds*, working draft (7 pages).

Using ideas introduced [15], we give a new proof of a theorem of Belolipetsky and Lubotzky on the isometry groups of real hyperbolic n -manifolds. The core idea for the trivial group can be seen in the proof of Proposition 2.4 in [15]. In that argument, we use a faithful action of a particular finite group on a character variety to produce asymmetric groups. Using a simple counting argument, we can upgrade this to realize any finite group as the symmetry group. The main difficulty is ensuring a certain finite group acts non-trivial on a character variety of the associated fundamental group for some reductive algebraic group. This finite group will be $[\mathrm{Comm}(\Gamma) : \Gamma]/\Gamma$ for a non-arithmetic lattice Γ . Using bending deformations for non-arithmetic manifolds constructed by Gromov–Piatetski-Shapiro, the faithfulness of this action can be arranged in the real hyperbolic setting where the reductive group is the isometry group of a higher dimensional hyperbolic space.

- (20) *Geodesic submanifolds of rank-1 arithmetic manifolds*, working draft (37 pages).

This article gives a detailed picture on the totally geodesic submanifolds of arithmetic, real rank one manifolds and orbifolds; real, complex, and quaternionic hyperbolic manifolds. The two basic problems addressed are the following. What arithmetic, real rank one manifolds M can a real rank one manifold N be isometrically immersed into? What do the primitive totally geodesic submanifolds of an arithmetic, real rank one manifold M look like? I show that the holomorphic totally geodesic submanifolds are more restricted than their Lagrangian counterparts. Similar results hold in the quaternionic hyperbolic setting. For example, there are precisely two commensurability classes of holomorphic totally geodesic $(n - 1)$ -dimensional complex hyperbolic submanifolds of an arithmetic complex hyperbolic n -manifolds derived from a hermitian form. Precise descriptions of the commensurability classes a manifold can be isometrically immersed into are given as well. In addition, finiteness results are shown. For example, the number of free homotopy classes of totally geodesic manifolds of a given model geometry, dimension, and of bounded complexity t is finite. The complexity is algebraic but is directly related to volume.

The main tools used are the theory of bilinear and hermitian forms over global fields and quaternion algebras, global class field theory, the geometry of local symmetric spaces, Tits' classification of arithmetic lattices, and classical Lie theory.

In preparation

- (21) (w/ A. Mohaddami) *Counting geodesic manifolds*, in preparation.

The purpose of this article is to initiate a study certain geometric counting functions on locally symmetric manifolds. A basic example of these functions is the function that counts the number of free isotopy classes of closed geodesics of length at most t . In this article, we study the asymptotic behavior of the function that counts totally geodesic surfaces in compact locally symmetric manifolds modeled on the symmetric spaces associated to the groups $\mathrm{SL}(3, \mathbf{R})$ or $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$; the functions are functions of a complexity proportional to volume. This work utilizes results and methods from a wide range of fields including Galois cohomology and counting points of bounded height on homogenous varieties.