Research Summary

My research to date touches a variety of different mathematical areas but can succinctly be summarized as geometric group theory and algebra with applications in geometry, topology, and spectral geometry. In what follows, I would like to elaborate further on some of my present research and provide a few new avenues for my research in the next few years. As my publication summary contains a detailed account of my past work, I will only speak on my present and future work here.

1 Present and future research

Though much of my work has had substantial geometric and topological flavors, I employ methods from several areas of math. With that said, many of my tools are algebraic; algebraic and analytic number theory, algebraic groups, algebraic geometry, finite group theory, involutions on algebras over local and global fields .etc. Therefore, it would not be unreasonable to describe me either as an algebraist doing geometry or as a geometric topologist who uses algebraic tools.

Every article I have written (past, present, and near future) addresses either geometric/topological problems for covers of a manifold or geometric/topological problems for submanifolds of a manifold. Both of these topics aim ultimately at producing a better understanding of the ambient manifold, either absolutely or in the context of a specified class of manifolds

1.1 Veech groups and Teichmüller curves

Translation surfaces give rise to affine groups whose corresponding subgroup in $\mathrm{SL}(2,\mathbf{R})$ is called a Veech group. Translation surfaces, Teichmüller curves, and their associated Veech groups have long interested topologists, dynamists, and geometers. The classification problem of Veech groups in $\mathrm{SL}(2,\mathbf{R})$ dates back at least to Thurston and remains an important open problem in this area (see [16] for instance for a survey on this area). At present, full classification of these special groups seems hopeless. Indeed, even the classification of Veech lattices, the case when the associated Veech group is a lattice in $\mathrm{SL}(2,\mathbf{R})$, is *very* open. In this section, I will discuss a few related problems and questions that aim at better understanding Veech lattices.

By Godement's compactness critierion, those Riemann surfaces commensurable with the modular curve are precisely the class of arithmetic, noncompact Riemann surfaces (see for instance [19]). It is well known (see either [20] or [26]) that a Veech lattice is necessarily noncocompact. In tandem these facts imply that if a Veech lattice is arithmetic, it must be commensurable with the modular group. An important special case of the classification problem of Veech lattices is the determination of those arithmetic Veech lattices. With Jordan Ellenberg [11], we made progress on this problem with the following result.

Theorem. If Γ is a finite index subgroup of $\mathrm{SL}(2,\mathbf{Z})$ that contains $\pm I_2$ and is contained in the level–2 principal congruence subgroup, then Γ is a Veech group.

Prior to this result, no class as large as this class had been known to be Veech (see [7], [25]). The condition on containing $\pm I_2$ is a mapping class lifting condition. The main improvement we further seek is the following:

Problem. If Γ is noncompact and arithmetic, then Γ is a Veech group.

The special case when Γ is maximal is a central case to consider and is already difficult. Using Borel's description of these groups [3], we know these groups are normalizers of certain finite index subgroups of $\mathrm{SL}(2,\mathbf{Z})$. Using this description, one must then construct a translation surface with this specified maximal group as it's symmetry group. We are afforded some leverage but it is unclear if it provides enough to build the translation surface. For example, by definition, these groups are symmetry groups both geometrically (via actions on builds/tress) and algebraically (normalizers of a specified subgroup). Moreover, by [11], there is a finite index subgroup of the maximal group that is a Veech group (in fact, we have infinitely many realizations for this finite index subgroup and infinitely many finite index subgroups we can use). Using one (or several) of these translation surfaces as a building block and the explicit description of the maximal group, we have a guide on how to explicitly construct the desired translation surface for which the maximal group is the associated Veech group. Note the associated Teichmüller curve in this case would necessarily be primitive (in both the geometric sense and the sense used in the theory of Teichmüller curves). It is worth mentioning that Gutkin–Judge [15] show that the associated translation surface must be tiled by parallelograms.

Reviewing [11], ones sees the possibility for a more general result on Veech groups. Specifically, our methods seem well suited for investigating the validity of the following statement:

Problem. If Γ is a Veech group and $\Delta < \Lambda$ is a finite index subgroup, then Δ is a Veech group.

This result in tandem with the realization of maximal arithmetic lattices (as discussed in the previous section), would yield a solution to the realization problem of noncocompact arithmetic lattices as Veech groups.

The basic idea for commensurability closure can be seen in [11]. We start with a translation surface S with associated Veech group Λ for which the pair produces a Teichmüller curve V in $\mathscr{M}_{g,n}$. Given a finite cover $f\colon \Sigma \to \Sigma_{g,n}$, we obtain a Teichmüller curve in $\mathscr{M}(\Sigma)$. By Gutkin–Judge [15], the associated Veech group Λ_f is commensurable with Λ. We select a family of covers $f_j\colon \Sigma \to \Sigma_{g,n}$ to ensure that $\Delta < \Lambda_{f_j}$. The philosophy is that for a generic selection, we have $\Delta = \Lambda_{f_j}$; this approach is taken in [1], [10], [11], [21], [22], and [12]. In the context of mapping class groups, Galois covers with abelian Galois group seem well suited because of certain lifting problem. In the case treated in [11], we utilized features particular to the initial Veech group; Λ was the level–2 congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$. Nevertheless, the philosophy is quite robust and I am confident this ideology can produce stronger results than [11]. It should be emphasized that these projects for Veech groups are ambitious but even small partial results on these projects would be quite nice since so little is known about Veech groups in general. For example, the following test case is unknown:

Conjecture. Let Λ be a Veech group and $\gamma \in \Lambda$ an element of infinite order. For each positive integer n, there exists a finite index subgroup Δ_n such that Δ_n is a Veech group and $\Delta_n \cap \langle \gamma \rangle = \langle \gamma^n \rangle$.

Using results from [18], one can arrange for $\Delta_n \cap \langle \gamma \rangle < \langle \gamma_n \rangle$. Utilizing some freedom in the construction of Δ_n, we hope to force equality generically.

1.2 Cohomological dimension spectra of groups

With Benson Farb, we have initiated the study of what we call the cohomological dimension spectrum of a finitely generated group with the following basic question as a guide.

Problem. For a fixed group Γ, what are the possible cohomological dimensions for subgroups Δ of Γ?

We define the *cohomological codimension* $\mathrm{ccd}_\Gamma(\Delta)$ of a subgroup Δ of Γ to be $\mathrm{cd}(\Gamma) - \mathrm{cd}(\Delta)$. Geometrically motivated, we study cohomological codimension for lattices Γ in $\mathrm{Sp}(n,1)$; our interest is more broadly in cocompact lattices Γ in real simple Lie groups. Specifically, we believe the following holds:

Conjecture. If $\Gamma < \mathrm{Sp}(n,1)$ is a cocompact lattice and $\Delta < \Gamma$ is a finitely generated, infinite index subgroup, then $\mathrm{ccd}_\Gamma(\Delta) \geq 4$.

Conjecture. $\mathrm{ccd}_\Gamma(\Delta) = 4$ if and only if is the (possibly deformed) image of the fundamental group of a totally geodesic quaternionic hyperbolic $(n-1)$–dimensional submanifold.

These conjectures would provide a new rigidity phenomenon, which have direct connections with the cohomology of lattices, dynamics of the geodesic flow, representations theory of real simple Lie groups, and spectral geometry of covers. For example, one potential application is the following.

Conjecture. If M be a cocompact quaternionic hyperbolic n–manifold, then both $H^2(M,\mathbf{R})$ and $H^3(M,\mathbf{R})$ are trivial.

Remark. It is worth noting that this vanishing conjecture is *not* a consequence on known vanishing theorems like Matsushima (see [4] for more on this) and thus would yield not only a strengthening of results like Matsushima but also demonstrate a new cohomological vanishing phenomenon. Moreover, the vanishing of $H^2(M)$ for a quaternionic hyperbolic n–manifold is expected. Indeed, these manifolds are conjectured to not be Kähler groups and it is also conjectured that Kähler groups should have non-trivial H^2. In addition, this cohomological vanishing would have representation theoretic applications (see [4]).

This application requires a result relating homological dimension and homology in the manifold M. This task can be reduced to a homological-homotopy optimization problem.

Problem. Let M be an closed, aspherical manifold. Given a non-trivial homology class $\alpha \in H_j(M)$, when does there exist a representation of this class by $f\colon Y \to M$ such that $f_*(\pi_1(Y))$ has infinite index in $\pi_1(M)$?

This question can also be cast into a lifting problem of the class to an infinite cover and can thus can be viewed as a homology separability problem. When M is negatively curved, one can try to use the positivity of simplicial volume. Geometric representatives whose simplicial volume is small relative to the geometry of M are potential candidates.

In [13], we provide evidence for Conjecture 2 by proving the following theorem.

Theorem. For a torsion free, cocompact lattice Γ of $\mathrm{Sp}(n,1)$ and an infinite index, finitely generated subgroup Δ of Γ, then $\mathrm{ccd}_\Gamma(\Delta) > 1$.

It is worth noting that this theorem is not true in general for cocompact lattices in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. In addition, this result is not a consequence of these lattices having Property (T), as there exist groups with Property (T) for which this gap theorem does not hold.

Broadly, one believes that the above should hold as a result of two separate theorems. First, M. Kapovich [17] proved for lattices in $\mathrm{SO}(n,1)$, that there is a direct relationship between the cohomological dimension $\mathrm{cd}(\Delta)$ of Δ and the critical exponent $\delta(\Delta)$ of Δ. Second, Corlette [9] proved that the critical exponent of Δ is either $4n+2$ or less than or equal to $4n$, where the former occurs if and only if Δ is finite index.

Recall, the critical exponent can be defined dynamically using Patterson–Sullivan measures and has an interpretation as a rate of mixing for the geodesic flow of X/Δ, where X is the associated symmetric space (this is the infinite cover associated to Δ if you wish). The critical exponent also has a spectral interpretation via harmonic measures and is directly related to λ_1 of the cover X/Δ, the smallest eigenvalue of the Laplacian acting on $L^2(X/\Delta)$. Since X/Δ comes equipped with an $\mathrm{Sp}(n,1)$–action, these small eigenvalues correspond to certain irreducible unitary representations of $\mathrm{Sp}(n,1)$; the trivial representation corresponds to case $\lambda_1 = 0$. Corlette's theorem is proved by showing certain complementary series representations of $\mathrm{Sp}(n,1)$ cannot occur.

Returning to our result, the above approach requires a generalization of Kapovich [17] relating critical exponent and cohomological dimension. The bulk of our work is in generalizing certain estimates of Besson–Courtios–Gallot on the p–Jacobian of a natural map in the real hyperbolic setting to the complex, quaternionic, and Cayley hyperbolic setting. We do need improvements on these estimates given in [2] in the real hyperbolic setting, as those estimates, which are applicable to the other settings, are insufficient to prove our results. The relationship to the critical exponent also appear in these volume estimates, as the critical exponent is also a factor in the change of volume formula.

Our present work is already very close to ruling out $4n-2$ as a possible cohomological dimension of a finitely generated subgroup. As the estimate of the p–Jacobian is fairly tight, the main quantity to control is the critical exponent. At present, with the added assumption that Δ is convex cocompact, we know that $\delta(\Delta) < 4n$. In our application, we require $\delta(\Delta) \leq 4n - \varepsilon$ for $\varepsilon > 0$ or a limiting argument to preclude clustering of the critical exponents of a family of finitely generated subgroups at $4n$. As none of these results utilize the fact that Δ is subarithmetic, using this condition might afford additional tools in ruling out certain complementary series representations and the desired ε–upgrade of [9] (or the obstruction to clustering). The cohomological dimension $4n-3$ seems substantially more difficult. It is worth mention that producing upper bounds for the critical exponent is extremely subtle as Corlette [9] has examples for increasing the critical exponent via deformations.

1.3 Asymptotic behavior of geometric counting functions

Studying the asymptotic behavior of counting functions arising from geometric considerations has been a basic and central topic in geometry for several years; growth rate of geodesics, flats .etc.

Problem. In a fixed locally symmetric manifold M, what are the asymptotic behaviors, as a function of volume, of the functions that count the number of totally geodesic submanifolds (of a fixed type) of bounded volume in M?

In [21], I produced asymptotic lower bounds for a counting function arising in spectral geometry. As a function of volume, the function counted the maximal number of pairwise non-isometric, isospectral locally symmetric manifolds there are with volume bounded above by t (see [8]). In addition to this work, Bou-Rabee and I [5, 6] investigated the asymptotic and average behavior of another class of geometric functions that measured the difficulty of finding a finite cover where a specified closed geodesic failed to lift.

In order to count totally geodesic submanifolds with a particular geometry, one must first parameterize them and then estimate those below the complexity threshold. As mentioned earlier, there exists a method for parameterizing the submanifolds though in practise it can be rather technical. Moreover, this method enumerates the commensurability classes via a Galois cohomology set and thus provides a rather indirect parametrization of the submanifolds.

With Mohammadi, in [23] we have begun the study of geometric counting functions. We have chosen two settings as tests cases for this topic.

- Hyperbolic surfaces in the arithmetic orbifold $X/\mathrm{SL}(3,\mathbf{Z})$, where $X = \mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3)$. This is directly related to counting rational quadratic forms on $\mathbf{Q}^3$.
- Hyperbolic surfaces in the Hilbert modular surface $\mathbf{H}^2 \times \mathbf{H}^2/\mathrm{PSL}(2,\mathbf{Z}[\sqrt{2}])$.

Abstractly, the commensurability classes of such surfaces can be described via representation theory and Galois cohomology. For example, in the second case, one has a map between the Brauer group over $\mathbf{Q}$ to the Brauer group over $\mathbf{Q}(\sqrt{2})$ and the classes that give rise to surfaces are those that map to the trivial element. Indeed, the commensurability rigidity results like in [24] can be described as the determination of certain algebraic forms of a classical group via an infinite collection of Galois cohomology classes. In general, each irreducible representation gives rise to a set of Galois cohomology classes that describes the commensurability classes of submanifolds arising from this representation. However, in both of the above cases, the representation theoretic aspect is trivial (there is only one representation of $\mathrm{SL}(2,\mathbf{R})$ in these groups up to conjugation). Having enumerated the commensurability classes of submanifolds, one then encounters a pair of problems. First, counting the growth of the surfaces arising from a fixed commensurability class using Gorodnik–Oh [14]. Second is the incorporation of the data from each class in the sum over all such classes. One of the technical issues is the dependence of error terms on the Galois cohomology class in [14].

Our long term goal is to investigate these functions for a wider class of manifolds. Unlike the case of closed geodesics, these counting functions can have an extremely wide range of behavior for manifolds even if one fixes the universal cover; the orbifold associated to $\mathrm{SL}(3,\mathbf{Z})$ has a plentiful supply of totally geodesic surfaces while other manifolds with the same model geometry have none. This wide range is seen by how varied the Galois cohomology sets can be. Indeed, for other manifolds modeled on $\mathbf{H}^2 \times \mathbf{H}^2$, explicit conditions on the class in the Brauer group representing the manifold can be written down for when the manifold has totally geodesic surfaces.

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