Research Summary

My research to date touches a variety of different mathematical areas but can succinctly be summarized as geometric group theory and algebra with applications in geometry, topology, and spectral geometry. In what follows, I would like to elaborate further on my past and present research, and provide a few new avenues for my research in the next few years.

1 Past work

With brevity in mind, I will briefly describe the main arcs in my past research, which fit into four topics. Though much of my work has had a substantial geometric and topological flavor, I employ methods from several areas of math. With that said, many of my tools are algebraic: algebraic and analytic number theory, algebraic groups, algebraic geometry, finite group theory, involutions on algebras over local and global fields. Therefore, it would not be unreasonable to describe me as an algebraist doing geometry or as a geometric topologist who uses algebraic tools. With full sincerity, my research is centered around solving problems and this has lead me in various directions with many collaborators.

**Geometric bounding** I will begin with a discussion of a problem in Riemannian geometry broadly called geometric bound. Given a closed, connected, smooth manifold $M$, we seek compact Riemannian manifolds $W$ with boundary $\partial W$ such that $M$ is diffeomorphic to a connected component of $\partial W$. The primary interest is on what Riemannian metrics on $M$ arise from the metrics induced from the manifolds $W$. Motivated by work of Gromov and conjectures of Farrell–Zdravkovska and Nimershiem, I investigated geometric bound for flat and almost flat manifolds $M$. One class of manifolds $W$ where flat manifolds naturally arise as boundary components are finite volume, complete, real hyperbolic orbifolds. To be precise, for noncompact, finite volume hyperbolic manifolds, one must neuter the manifold in order to produce a compact manifold with boundary. Nimershiem proved that every flat 2 and 3 manifold arises as a boundary component from manifold $W$; Hamrick–Royster showed flat manifolds topologically bound 16 years before. In fact, she proved a density result on the possible metrics that occur on a fixed flat manifold in these dimensions. Long–Reid in a pair of papers provided a counterexample to one part of Farrell–Zdravkovska conjecture and showed that every flat manifold arises as the boundary component of an arithmetic hyperbolic orbifold. I strengthened and generalized of the work of Nimershiem and Long–Reid. The totality of this work classified the possible metrics on the boundary components of noncompact, arithmetic, real rank–1 manifolds. This work proved a stronger form of Nimershiem’s conjecture and answered questions of Hirzebruch and Long–Reid. I also produced counterexamples to a generalization of the Farrell–Zdravkovska conjecture for Sol 3–manifolds. Though I resolved many questions in this area, the full Farrell–Zdravkovska conjecture still remains open.

This work involved geometric and arithmetic results. However, some of the tools required were new results on the profinite topology of arithmetic lattices. The most elementary to state of these results was a generalization of Selberg’s lemma on the existence of torsion free, finite index sub-
group of finitely generated linear groups. Specifically, I proved that one can select the torsion free subgroup to also contain certain finitely generated, torsion free subgroups.

Some additional tools used were group actions on character varieties, structure theory nilpotent Lie groups, Lie algebras, their affine groups, and cocompact lattices, $L$–functions, and $p$–adic Lie groups.

Spectral geometry Broadly, the subject of spectral geometry investigates eigenvalue spectra for differential operators $A$ on manifolds $M$ with the prime example being the eigenvalue spectrum of the Laplace operator $\Delta$ acting on $L^2(M)$ or $k$–forms; in operator theory, inverse spectral problems have been around for at least 70 years. Milnor produced the first examples of non-isometric manifolds with identical Laplace spectrum. In the nearly fifty years since, there have been many new constructions and contributions (too numerous to mention here). Of most relevance for our discussion was a construction due to Sunada (motivated by investigations of zeta functions of number fields), which provides a general method for constructing isospectral manifolds. Building primarily on Sunada but also on work of Spatzier and of Brooks–Gornet–Gustafson, I produced many new non-isometric examples of locally symmetric, compact Riemannian manifolds $M,N$ with identical Laplace spectra. This work answered in the affirmative an implicit question of Spatzier. In addition, I produced what I called isospectral towers on locally symmetric compact manifolds, which were only previously known to exist for a few classes of symmetric spaces by work of Vigcnas and Lubotzky–Samuel–Vishne. Aside from Sunada’s method, I employed results on from the theory of algebraic groups, arithmetic lattices, finite groups, and analytic number theory. The main idea was a random strategy for producing candidates that, via work of Belolipetsky–Lubotzky and a counting argument, were shown to ”generically” be non-isometric. I also generalized, to all symmetric spaces with non-compact simple isometry group, lower bounds on an isospectral counting function introduced by Brooks–Gornet–Gustafson for hyperbolic 2–space.

With Chris Leininger, Walter Neumann, and Alan Reid, we investigated the dependence of multiplicities in the Laplace spectrum. Via a weakening of Sunada’s method, we showed that some of the so-called heat invariants (like volume and total scalar curvature) depend on the multiplicities of the eigenvalues. It is well known that the Laplace eigenvalue spectrum has strong ties with the geodesic length spectrum $\mathcal{L}(M)$ comprised of lengths of geodesic geodesics counted with multiplicity (via the heat, Schrödinger, and wave equations). We also showed these heat invariants depend on the multiplicities in the geodesic length spectrum. The main tools used were finite group theory, analytic and algebraic number theory, arithmetic lattices, representation theory, and $p$–adic Lie groups.

With Alan Reid, we initiated the study of higher dimension geometric spectra, which generalizes the study of the geodesic length spectrum. For hyperbolic 3–manifolds and totally geodesic sub–surfaces, we proved commensurability class rigidity of the so-called genus spectrum for arithmetic manifolds containing a totally geodesic surface. We also generalized Sunada’s construction to this setting and produced many interesting examples of pairs with identical totally geodesic surfaces. The main tools used were analytic and algebraic number theory, quaternion algebras, and finite group theory. This subject can also be cast purely in terms of algebraic groups and their algebraic subgroups but for length constraints, we omit this interesting detour.
Mapping class groups and Teichmüller theory  My work with mapping class groups and moduli space has largely been concerned with the congruence topology on mapping class groups, and algebraic and geometric connections of this topology to moduli space. My first paper on this topic was with Chris Leininger, where we investigated the congruence topology via mapping class groups actions on character varieties and the so-called Johnson filtration. We introduced a natural family of subgroups of mapping class groups that behaves like totally geodesic subgroups and showed many previously known subgroups arise this way. The methods we used were a mixture of algebraic and topological; some of the ideas have their roots in work of Bass–Lubotzky on automorphism groups of schemes of finite type.

Following a short outline by Thurston, I gave an elementary proof that pure braid groups, which are essentially mapping class groups of punctured spheres, have the congruence subgroup property. This result was first shown by Diaz–Donagi–Harbater and subsequently by Asada and more recently Boggi. However, all of these works utilize algebro-geometric and field theoretic approaches, while our approach relies on elementary group theory and geometry.

With Jordan Ellenberg, we proved a strong form of the congruence subgroup property for mapping class groups of punctured tori, improving a result of Asada—an elementary proof of a quantitative version of Asada’s theorem has also recently been given by Bux–Ershov–Rapinchuk. The main corollaries of our work are a converse to a theorem of McMullen on the algebraic structure of Teichmüller curves, progress on the Thurston’s classification problem for Veech subgroups of $\text{SL}(2, \mathbb{R})$, and a strengthening of a theorem of Möller, which shows that $\text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of isomorphism classes of Teichmüller curves. This work has very recently been used by Avila–Matheus–Yoccoz to show the existence of Teichmüller curves with complementary series.

Subgroup growth and number theoretic results on groups  In two papers with Khalid Bou-Rabee, we investigated ties between subgroup growth, word growth, quantitative residual problems, and number theoretic results on linear groups. Part of this topic was initiated in a paper of Bou-Rabee, where he focused on quantitative residual problems. The main players in this study are functions called divisibility functions (associated to a family of finite index subgroups) that measure how hard it is to construct a finite quotient where a fixed non-trivial word has non-trivial image. We established basic inequalities that relate functional norms of divisibility functions to subgroup and word growth functions (and residual girth). We also established lower and upper bounds on the $L^\infty$–norm of divisibility functions. The main tools used were methods in combinatorial group theory, analytic and algebraic number theory, and covering space theory. Our first paper and part of Bou-Rabee’s article have some connections to recent work by Hadad and Rivin. One of our main results answers Problem 15.35 in the Kourovka notebook asked by Oleg Bogopolski. Our second paper addressed the $L^1$–norm of divisibility functions, where we proved the finiteness of $L^1$–norms for finitely generated linear groups over $\mathbb{C}$. This result might seem like a specialized result but is intimately connected to subgroup growth functions on groups and a generalization of Bertrand’s postulate on the distribution of primes to finitely generated linear groups over $\mathbb{C}$. Indeed, we prove two versions of Bertrand’s postulate for this class of groups, the proof of the second version avoids the use of the Strong Approximation Theorem. The main tools employed in this work are profinite groups, analytic and algebraic number theory, $p$–adic Lie groups, elementary group theory, and involved counting arguments. Ties to classical number
theory extend beyond just the generalization of Bertrand’s postulate, as we have (to the best of our knowledge) found new results on Wieferich primes and their generalization as a consequence of trying prove some of the above results.

2 Present research

In this short section, I will describe two projects for which most of the core mathematics has been completed.

Isometry groups and lattices  As a result of my collaboration with Jordan Ellenberg on mapping class groups, we have recently found a new proof of a theorem of Belolipetsky–Lubotzky on determining what finite groups occur as isometry groups of a closed hyperbolic $n$–manifold $M$. This problem can be thought of as an inverse Galois problem for isometry groups, where the quotient by the isometric action plays the role of $\mathbb{Q}$ and the manifold $M$ plays the role of the Galois extension. Our proof involves group actions on character and representation varieties for finitely presented groups, the Weil conjectures (or alternatively, the Strong Approximation Theorem), finite group theory, and counting arguments. Moreover, our method seems to generalize to settings that Belolipetsky–Lubotzky could not handle; if this is the case, our work will answer a question from the article Belolipetsky–Lubotzky.

Dynamics and convex geometry  Recently with Benjamin Schmidt, we have investigated recognition problems for point X-rays of convex bodies in $\mathbb{R}^2$ (and more generally $\mathbb{R}^n$). Loosely, given a convex body $B$ and a point $p \not\in B$, a point X-ray of $B$ from $p$ is a function $X_{p,B} : [0,2\pi) \to \mathbb{R}$. The value at an angle $\theta$ is the length of the line segment determined by intersecting the directed ray emanating from $p$ with direction $\theta$ with the convex body $B$. Recognition problems ask if bodies can be determined from various X-ray data. The problems we address originated with Hammer in the early 1960’s. At present, we can resolve the strong recognition problem for smooth convex bodies with three point X-rays and have finiteness results for smooth bodies with two point X-rays—the strongest possible result is strong recognition for two point X-rays. The strong recognition for $k$ points asserts that for any $k$ distinct points and any convex body $B$, the body $B$ is uniquely determined by these $k$ X-rays. The case of $k = 2, 3$ improves work of Falconer, Gardner, and Volcic. Our methods are a blend of group theory, elementary topology, and geometry, but mainly rely on the dynamics of certain self-homeomorphisms of $S^1$ and $D^2$. One novelty of our approach is that our methods work with general densities on $\mathbb{R}^2$.

3 Future research

I will end with a list topics where either I have on-going projects, on-going discussions, or on-going interest. It goes without saying that the above topics will continue to hold my interest and I expect to continue contributing to those subjects. In addition, the topic below do not represent all of the additional directions for my future research.
• With Bou-Rabee, we plan on further investigating the ties between subgroup growth, word growth, divisibility functions, residual averages, and profinite groups. Tools from Fourier analysis seem relevant to understanding divisibility functions and residual averages, and we hope to further push the ties from analytic number theory to general linear groups. We also want to investigate profinite groups and their subgroups; versions of Grothendieck’s conjecture, analytic behavior of divisibility functions on profinite groups. We would also like to extend the polynomial upper bounds of Bou-Rabee for the $L^\infty$–norm of normal divisibility functions to general finitely generated linear groups; this is likely to involve a study of representation complexity versus word complexity, algebraic geometry, and number theory. There are numerous questions that have come from this work with many possible connections to several areas.

• Higher geometric spectral rigidity. With Carolyn Gordon and David Webb, we have some preliminary results for flat tori and hope to extend them to infranil manifolds and certain arithmetic locally symmetric manifolds. Even the case of commensurability could be interesting with regard to the geometric rigidity the totally geodesic submanifolds; for instance, incommensurable isospectral manifolds. In addition to understanding ties between the geometry of a manifold and the geometry of its submanifolds, we would like count submanifolds as a function of volume. Are there any trace-type formulas for submanifolds?

• Classifying totally geodesic submanifolds of arithmetic manifolds. In an unpublished paper, I have already addressed this problem for arithmetic real rank–1 manifolds. With Skip Garibaldi and Dave Witte-Morris, we hope to address a broader class of manifolds and submanifolds. A related problem is understanding where a fixed manifold can be mapped as primitive totally geodesic embedded or immersed submanifold. There are several interesting questions one can pose here and to my knowledge, little work has been done on this very natural problem.

• Commensurability invariants for Zariski dense subgroups of algebraic groups. With Skip Garibaldi, we will investigate this problem first for subgroups of $SU(2, 1)$. We have applications for this work on geodesics in complex hyperbolic 2–manifolds in mind. There are several interesting computational problems in geometry and algebra that arise here. For example determining a cyclic algebra (possibly with an involution) from a finite collection of elements. The tools we develop with likely be useful in studying other problems in this area.

• Cohomology arising from submanifolds of locally symmetric manifolds. This topic is the source for three projects. Two involve Jean Lafont and Benjamin Schmidt, while one involves Matthew Stover. The project with Stover has ties to Galois representations and cohomology of PEL modular varieties. At the core of these projects are the basic problems of relating cohomology on submanifolds to cohomology on the ambient manifold and relating cohomology classes to the geometry of the manifold. We have a general conjecture about a specific relationship between cohomology on submanifolds and the virtual extension of these classes to the ambient manifold, when both are locally symmetric and finite volume. Again, the case of lattices in $SU(n, 1)$ is a nice launching point to test our methods and conjectures.
• Geometry of locally symmetric and locally homogenous manifolds. With Misha Belolipetsky, we plan on investigating isometry groups of arithmetic manifolds. For the realization problem of isometry groups of higher rank locally symmetric manifolds, it is unknown in general if the trivial group can be realized. We would like to solve this problem and have some ideas. With Misha Belolipetsky and Benson Farb, we are investigating the construction of lattices in linear algebraic groups. The construction of irreducible non-cocompact and cocompact lattices in Lie subgroups of affine groups of nilpotent and solvable Lie groups is classical and not completely understood; for instance $\mathbb{R}^n \rtimes \phi \text{SL}(m, \mathbb{R})$. A fair amount of work has already been put into this problem and a draft of an article exists.

• With Jeremy Kahn, we want to investigate spectral and geodesic geometry of surfaces. For instance, we would like to investigate when isospectral Riemann surfaces are commensurable. It might be possible to employ recent work of Bowen and Kahn–Markovic to this question. We hope at the least to better understand the relationship between isospectrality and commensurability for Riemann surfaces. We would also like to better understand geodesic length coincidence sets in the moduli space of hyperbolic metrics for finite sets of simple closed curves. These are metrics where the curves have geodesic representatives of equal length. The case of two simple curves is already interesting and we have a modest goal of understanding these sets at infinity.

• With Dubi Kelmer, we would like to extend some of the work of Kelmer–Sarnak on strong spectral gaps for compact quotients of products of simple Lie groups. The existence of such gaps can be viewed as a weak version of the congruence subgroup property for the associated lattices. Lattices arising from cyclic division algebras equipped with a hermitian involution are rather mysterious and provide a good class (for many reasons) to start with. Some of the core work required for this project is required for a few of the above projects, namely those involving lattices in unitary groups. It should be noted that for the classes of lattices we are investigating, the congruence subgroup problem (Serre’s conjecture) has not been resolved.