

# EVERY 1-GENERIC COMPUTES A PROPERLY 1-GENERIC

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ABSTRACT. A real is called *properly*  $n$ -generic if it is  $n$ -generic but not  $n + 1$ -generic. We show that every 1-generic real computes a properly 1-generic real. On the other hand, if  $m > n \geq 2$  then an  $m$ -generic real cannot compute a properly  $n$ -generic real.

## 1. INTRODUCTION

The notions of measure and category (or in forcing terminology, random (Solovay) and Cohen forcing) have made their way into computability theory via the notions of restricted randomness and genericity. For Cohen reals this was done by Jockusch [Joc80], who studied  *$n$ -genericity*, that is, genericity where the forcing relation is restricted to  $n$ -quantifier arithmetic (as Jockusch and Posner [JP78] observed, a real is  $n$ -generic iff for all  $\Sigma_n^0$  sets of strings  $S$ , there is some initial segment  $\sigma$  of  $A$  such that  $\sigma \in S$  or  $\sigma \not\leq \tau$  for all  $\tau \in S$ .) Restricted genericity gives rise to a proper hierarchy (every  $n + 1$ -generic real is also  $n$ -generic but not vice-versa). Thus, we can define a real to be *properly*  $n$ -generic iff it is  $n$ -generic and not  $n + 1$ -generic. [A related notion, first discussed by Kurtz [Kur81], is that of *weak*  $n$ -genericity. Here a real  $A$  is weakly  $n$ -generic iff  $A$  meets all dense  $\Sigma_n^0$  sets of strings. Kurtz [Kur81] showed that weak genericity refines the genericity hierarchy, with  $n$ -generic  $\subsetneq$  weakly  $n + 1$ -generic  $\subsetneq$   $n + 1$ -generic.]

The study of reals random at various levels of the arithmetical hierarchy was introduced by Martin-Löf [ML66]. A real  $A$  is called  *$n$ -random* iff for all  $\Sigma_n^0$ -tests  $\{U_n : n \in \mathbb{N}\}$ , we have  $A \notin \bigcap_n U_n$ . Here, a  $\Sigma_n^0$ -test is a (uniform) collection of  $\Sigma_n^0$ -classes  $\{U_n : n \in \mathbb{N}\}$ , such that  $\mu(U_n) \leq 2^{-n}$ , where  $\mu$  is Lebesgue measure. (We refer the reader to Downey, Hirschfeldt, Nies and Terwijn [DHNT] for a general introduction to results relating genericity, randomness and relative computability, as well as to the forthcoming books Nies [Nie] and Downey and Hirschfeldt [DH].)

Both  $n$ -genericity and  $n$ -randomness can be relativized to a given real  $Z$  by replacing  $\Sigma_n^0$  objects by ones that are  $\Sigma_n^0$  relative to  $Z$ . For instance, a real  $A$  is  $n$ -random over  $Z$  iff  $A \notin \bigcap_n U_n$  for all tests  $\{U_n : n \in \mathbb{N}\}$  that are  $\Sigma_n^0$  relative to  $Z$ . It is easy to see that a real is  $n$ -generic iff it is 1-generic over  $\emptyset^{(n-1)}$ . Kurtz [Kur81] showed that this is also true for randomness; that is, a real is  $n$ -random iff it is 1-random over  $\emptyset^{(n-1)}$ .

There are striking similarities between the ways these two notions interact with Turing reducibility. For example, relatively 1-generic reals form minimal pairs, as

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do relatively 2-random reals. Another nice example is van Lambalgen's Theorem (van Lambalgen [VL87]) which says that  $A \oplus B$  is  $n$ -random iff  $A$  is  $n$ -random and  $B$  is  $n$ -random over  $A$ ; Yu [Yu] proved the analogous statement for genericity.

There are interesting distinctions as well. For example, there are complete  $\Delta_2^0$  1-random reals, whereas all 1-generic reals are generalised low.

This paper is motivated by a result of Miller and Yu [MY]:

**Theorem 1.1** (Miller and Yu). *Let  $A$  be 1-random over a real  $Z$ , and let  $B$  be 1-random and computable in  $A$ . Then  $B$  is 1-random over  $Z$ .*

In particular, letting  $Z = \emptyset^{(n-1)}$ , if  $A$  is  $n$ -random and  $B \leq_T A$ , with  $B$  1-random, then  $B$  is  $n$ -random. Asking whether the same property holds for Cohen genericity yields both a similarity and a distinction from the random case. We will show that the analogue of Miller and Yu's result holds in the generic case, if the bottom real  $B$  is 2-generic:

**Theorem 1.2.** *Let  $A$  be 1-generic over a real  $Z$ , and let  $B$  be 2-generic and computable in  $A$ . Then  $B$  is 1-generic over  $Z$ .*

As a result, it is impossible for, say, a 3-generic real to compute a properly 2-generic real. We mention that Theorem 1.2 may be known, but is not yet found in print, and so we include a proof here.

On the other hand, the analogue of Miller and Yu's result *always* fails when 2-genericity is reduced to 1-genericity:

**Theorem 1.3.** *Every 1-generic real computes a properly 1-generic real.*

In fact we prove something somewhat stronger.

**Theorem 1.4.** *Every 1-generic real computes a 1-generic real that is not weakly 2-generic.*

We mention some related results: Haught [Hau86] showed that below  $\mathbf{0}'$ , the 1-generic degrees are downward closed; Martin showed that for  $n \geq 2$ , the  $n$ -generic degrees are downward dense (see [Joc80]). More such results are surveyed in [GM03], which gives some applications.

Several questions remain:

**Question 1.5.** *Can a sufficiently generic real compute a weakly 2-generic real that is not 2-generic? Must it?*

A degree is properly 1-generic if it contains a 1-generic real but no 2-generic real.

**Question 1.6.** *Can a sufficiently generic real compute a properly 1-generic Turing degree? Must it?*

**1.1. Notation and terminology.** We work with Cantor space  $2^\omega$ . A *class* is a subset of  $2^\omega$ . For every  $\sigma \in 2^{<\omega}$ , let  $[\sigma]$  denote the clopen interval in  $2^\omega$  defined by  $\sigma$ , i.e.,  $\{X \in 2^\omega : \sigma \subset X\}$ . For any  $W \subseteq 2^{<\omega}$ , we let  $\mathcal{W} = \bigcup_{\sigma \in W} [\sigma]$  be the open class defined by  $W$ . An open class  $\mathcal{O} \subseteq 2^\omega$  is *enumerable* by some Turing degree  $\mathbf{b}$  (we write that  $\mathcal{O}$  is  $\Sigma_1^0(\mathbf{b})$ ) if it is defined by some  $W$  that is computably enumerable by  $\mathbf{b}$ . (So a real  $X \in 2^\omega$  is weakly 2-generic iff it is an element of every dense, open class that is enumerable by  $\mathbf{0}'$ .) We say that  $\mathcal{O}$  is *c.e.* if it is enumerable by  $\mathbf{0}$ . This terminology can be used up the arithmetic hierarchy; thus a  $\Pi_1^0(\mathbf{b})$  class is the complement of an open set enumerable by  $\mathbf{b}$  (equivalently, the

set of paths through a tree computable by  $\mathbf{b}$ ); and a  $\Pi_2^0(\mathbf{b})$  class is the intersection of a countable sequence of open sets, uniformly enumerable by  $\mathbf{b}$ .

If  $W \subseteq 2^{<\omega}$ , we say that a string  $\sigma \in 2^{<\omega}$  *decides* (or *forces*)  $W$  if either  $\sigma \in W$  or no extension of  $\sigma$  is in  $W$ . Topologically, either  $[\sigma] \subseteq W$  or  $[\sigma]$  is a subset of the complement of  $W$ . So  $X \in 2^\omega$  is 1-generic iff for every c.e.  $W \subseteq 2^{<\omega}$ , the real  $X$  lies in some interval  $[\sigma]$  that decides  $W$ .

Turing functionals are codes of partial computable functions from  $2^\omega$  to  $2^\omega$ . Formally, a Turing functional is a c.e. set  $\Phi \subseteq 2^{<\omega} \times 2^{<\omega}$  that is *consistent*: for  $\sigma' \subseteq \sigma$ , if  $(\sigma', \tau') \in \Phi$  and  $(\sigma, \tau) \in \Phi$ , then  $\tau' \subseteq \tau$ . For  $\sigma \in 2^{\leq\omega}$ , let  $\Phi(\sigma) = \bigcup\{\tau : \exists \sigma' \subseteq \sigma [(\sigma', \tau) \in \Phi]\}$ .

As usual, during a construction, at each stage, all expressions involving dynamic objects are evaluated according to the state of the objects (either constructed or given) at the stage. Usual conventions apply; if during a construction we approximate a  $\Delta_2^0$  set  $A$ , then the value of  $A$  on every  $x$  is carried over from one stage to the next unless we explicitly act to change that value.

If  $\sigma, \tau \in 2^{<\omega}$  then  $\sigma\tau$  denotes the concatenation of  $\sigma$  and  $\tau$ . A digit  $i \in \{0, 1\}$  often stands for the string  $\langle i \rangle$ . If  $\sigma \in 2^{<\omega}$  and  $k < \omega$  then  $\sigma^k$  is the concatenation of  $\sigma$  with itself  $k$  times.

We let  $W_0, W_1, \dots$ , be a uniform enumeration of all c.e. subsets of  $2^{<\omega}$ . The enumeration is arranged so that at every stage  $s > 0$ , there is exactly one string  $\sigma$  and one  $e$  such that  $\sigma$  enters  $W_e$  at stage  $s$ . We also assume that if  $e \geq s$  then  $W_e$  is empty at stage  $s$ .

## 2. A POSITIVE RESULT

We prove Theorem 1.2: Let  $A$  be 1-generic over a real  $Z$ , and let  $B$  be 2-generic and computable in  $A$ . Then  $B$  is 1-generic over  $Z$ .

*Proof of Theorem 1.2.* Let  $A$  be 1-generic over  $Z$  and let  $B \leq_T A$  be 2-generic. Let  $\Phi$  be a Turing functional such that  $\Phi(A) = B$ . Let  $W \subseteq 2^{<\omega}$  be c.e. in  $Z$ ; we may assume that  $W$  is closed upwards.

Suppose that  $B \notin W$ . Let  $\tilde{W} = \{\sigma \in 2^{<\omega} : \Phi(\sigma) \in W\}$ . Certainly  $\tilde{W}$  is c.e. in  $Z$ . Since  $A \notin \tilde{W}$ , we know that there is some  $\sigma^* \subset A$  with no extension in  $\tilde{W}$ .

Let  $U = \{\tau \in 2^{<\omega} : \neg \exists \sigma \supseteq \sigma^* [\tau \subseteq \Phi(\sigma)]\}$ . The set  $U$  is co-c.e., and  $B \notin U$ , so there is some  $\tau^* \subset B$  with no extension in  $U$ . Thus if  $\tau \supseteq \tau^*$  then there is a  $\sigma \supseteq \sigma^*$  such that  $\tau \subseteq \Phi(\sigma)$ . Since  $\sigma \notin \tilde{W}$ , we have  $\Phi(\sigma) \notin W$ . Since  $W$  is closed upwards,  $\tau \notin W$ . Thus  $\tau^*$  has no extension in  $W$ .  $\square$

## 3. COMPUTING PROPERLY 1-GENERIC SETS

In this section we prove theorem 1.4: Every 1-generic real  $X$  computes a 1-generic real that is not weakly 2-generic.

To do this, we construct a Turing functional  $\Gamma$  with the following properties:

- (1) There is a dense  $\Pi_2^0(\mathbf{O}')$  class that is contained in the domain of  $\Gamma$ , and whose image under  $\Gamma$  consists of 1-generic sets.
- (2) The range of  $\Gamma$  is contained in a nowhere dense  $\Pi_1^0(\mathbf{O}')$  class.

To see that this suffices, assume that  $X$  is weakly 2-generic (if  $X$  is not weakly 2-generic then we are of course done). Let  $\mathfrak{A}$  be the class guaranteed by (1). The real  $X$  is an element of any dense  $\Sigma_1^0(\mathbf{O}')$  class, hence of any countable intersection of such classes (recall Baire's theorem that states that an intersection of countably

many dense, open sets is dense); so  $X \in \mathfrak{A}$ . Then  $\Gamma(X)$  (which is computable by  $X$ ) is 1-generic by (1), and is not weakly 2-generic because by (2), it misses a dense open set enumerable by  $\mathbf{0}'$ .

### 3.1. Discussion.

3.1.1. *Getting property 1.* To make the image of  $\mathfrak{A}$  under  $\Gamma$  consist of 1-generic sets, for each  $e < \omega$ , we must construct a dense, open set  $\mathcal{S}_e$  such that for all  $\sigma \in \mathcal{S}_e$ ,  $\Gamma(\sigma)$  decides  $W_e$ ; and further we must make the sequence  $\mathcal{S}_0, \mathcal{S}_1, \dots$  uniformly enumerable by  $\mathbf{0}'$ , so that we can define  $\mathfrak{A} = \bigcap_e \mathcal{S}_e$ .

Consider  $W_0$ . A simple plan for meeting it would be setting  $\mathcal{S}_0 = 2^\omega$  and acting as follows: if  $W_0$  is empty, do nothing; if there is some  $\tau \in W_0$ , let  $\Gamma(\langle \rangle) = \tau$ . Now move to  $W_1$ . Of course, this plan is not effective, so we must use the priority method for our construction. Again, a naïve approach would be as follows: While  $W_0$  is empty, do nothing, and let weaker requirements  $(W_1, W_2, \dots)$  act if they want to. If some string  $\tau$  enters  $W_0$  then injure the weaker requirements and set  $\Gamma(\langle \rangle) = \tau$ . The problem here is that we cannot cancel the axioms that our work for  $W_1, W_2, \dots$  had us enumerating into  $\Gamma$ , so if we want to keep  $\Gamma$  consistent, we cannot make the definition we like. The solution is to break up the playing ground into pieces, let weaker requirements work on some of the pieces, and make sure that there is sufficient room for the stronger requirement to act if necessary.

Here is the strategy for  $W_0$ . In the beginning, we mark the interval  $2^\omega = [\langle \rangle]$  to work on  $W_0$ . We break the interval up into infinitely many disjoint subintervals whose union is dense in  $2^\omega$ , say  $[1], [01], [001], [0001], \dots$ . For the time being, each such subinterval believes it has met the  $W_0$ -requirement by forcing its image under  $\Gamma$  into the complement of  $W_0$ , simply because  $W_0$  is still empty. So we can be generous and let each subinterval work for the next requirement  $W_1$ .

At a later stage, some string  $\tau$  enters  $W_0$ . Only finitely many subintervals have been spoiled for  $W_0$ ; so we can define  $\Gamma$  to be  $\tau$  everywhere else. On the spoiled intervals, we need to work again for  $W_0$ ; since definitions of  $\Gamma$  have been made on possibly small subsubintervals, we need to break the spoiled region into small intervals on which we individually work on  $W_0$ .

We let  $\mathcal{S}_0$  be the collection of intervals  $\sigma$  that are “good” for  $W_0$ , which are those intervals on which we ensure that  $\Gamma(\sigma)$  meets  $W_0$ , and those at which we had a correct belief that  $[\Gamma(\sigma)] \cap W_0 = \emptyset$ . This set will in fact be d.c.e., and so certainly  $\Sigma_2^0$ ; and it satisfies the  $W_0$  requirement. Since we break it up into finer and finer subintervals (each time our hopes for an easy win are dashed),  $\mathcal{S}_0$  ends up being dense.

The strategy for weaker  $W_e$  is similar, except that of course we need to take into consideration injury by stronger requirements.

3.1.2. *Getting property 2.* To ensure that the range of  $\Gamma$  is nowhere dense, we could, whenever we define some axiom  $\Gamma(\sigma) = \tau$ , pick some extension  $\tau'$  of  $\tau$  and declare that no value of  $\Gamma$  may ever extend  $\tau'$ . This straightforward approach, however, interferes with the priority mechanism that ensures property (1), in the following way. Suppose that we mark some interval  $[\sigma_0]$  for  $W_1$ , and later define  $\Gamma(\sigma_1) = \tau$  for some  $\sigma_1 \supset \sigma_0$ , marking  $[\sigma_1]$  for  $W_2$ . We then declare that the range of  $\Gamma$  must be disjoint from  $[\rho]$ , where  $\rho \supset \tau$ . A later  $W_0$  action elsewhere invalidates  $[\sigma_0]$ 's marking, so we mark  $[\sigma_1]$  for  $W_0$ . Then, some string extending  $\rho$  enters  $W_0$ , but  $W_0$  is prohibited from winning by directing  $\Gamma$  through that string on a

subinterval of  $[\sigma_1]$ . We will indeed direct  $\Gamma$  to go through some extension  $\tau'$  of  $\tau$  that is incomparable with  $\rho$ , and this presumably will give us another chance of attacking  $W_0$ ; but this process may repeat itself, since following the straightforward approach compels us to first declare some extension  $\rho'$  of  $\tau'$  disjoint from the range of  $\Gamma$ . After infinitely many failed attempts at meeting  $W_0$  we have a real in the range of  $\Gamma$  belonging to the closure of  $\mathcal{W}_0$  but not to  $\mathcal{W}_0$  itself.

This in fact *must* happen, because we made the collection of prohibited intervals a dense c.e. class, ensuring that no element of the range of  $\Gamma$  is even weakly 1-generic (indeed, the recursion theorem and the “slowdown lemma” imply that there is some  $e$  such that  $W_e$  is the set of prohibited intervals, and that every  $\sigma$  is enumerated into  $W_e$  only after it was declared prohibited). The solution is to use the priority mechanism that was introduced for getting property (1). When we define  $\Gamma(\sigma) = \tau$  for meeting  $W_e$ , we define one extension to be prohibited with priority  $e$ . This prohibition can be ignored by strings  $\sigma' \subset \sigma$  that are working for stronger  $W_{e'}$ . The whole mechanism does the work for us, so we in fact do not need to use the word “prohibited” during the construction, just to make  $\Gamma(\sigma)$  long enough.

**3.2. Construction.** Here is the formal construction.

In the beginning, the entire space  $2^\omega = [(\langle \rangle)]$  is marked *active* for  $W_0$ . (When we mark a clopen set  $[\sigma]$  active for some  $W_e$ , we also say that the string  $\sigma$  is marked active for  $W_e$ .)

At *stage*  $s$ :

**1.** A string  $\tau$  is enumerated into some  $W_e$ . Suppose that there is some  $\sigma$ , marked active for  $W_e$ , such that  $\Gamma(\sigma) \subseteq \tau$  (there will be at most one such  $\sigma$ ). Do the following:

- Unmark  $[\sigma]$  from being active for  $W_e$ . Choose a very large number  $m$ .
- Enumerate  $(\sigma 0^s, \tau 1^m)$  into  $\Gamma$ . Mark  $[\sigma 0^s]$  as active for  $W_{e+1}$ .
- For every  $k < s$ , remove all markings of strings  $\sigma' \supseteq \sigma 0^k 1$ . For every  $\sigma' \in 2^m$  extending  $\sigma 0^k 1$  for some  $k < s$ , mark  $[\sigma']$  as active for  $W_e$ ; find some  $\tau_{\sigma'} \supset \Gamma(\sigma')$  of length  $m$  that is incompatible with  $\tau 1^m$ , and enumerate  $(\sigma', \tau_{\sigma'} \sigma' 1)$  into  $\Gamma$ .

**2.** Inductively for  $e < s$ , for every  $[\sigma]$  that is marked as active for  $W_e$ , for  $k \leq s$ , if  $[\sigma 0^k 1]$  is not yet marked, then mark it as active for  $W_{e+1}$  and enumerate  $(\sigma 0^k 1, \Gamma(\sigma) 0^k 11)$  into  $\Gamma$ .

This completes the construction.

**3.3. Verification.** For  $e < \omega$ , we define sets  $S_e^+$  and  $S_e^-$  as follows, and let  $S_e = S_e^+ \cup S_e^-$ .

- (1) Suppose that at stage  $s$ , an interval  $[\sigma]$  is marked active for  $W_{e+1}$ . If the last digit of  $\sigma$  is 1 then we enumerate  $\sigma$  into  $S_e^-$ . Otherwise, we enumerate it into  $S_e^+$ .
- (2) If  $\sigma \in S_e^-$  but at some stage  $t$ , the  $W_{e+1}$ -mark is removed from  $\sigma$ , then  $\sigma$  is extracted from  $S_e^-$ .
- (3) If  $\sigma \in S_e^+$  but at some stage  $t$ , action is taken for some  $\sigma' \subset \sigma$ , then  $\sigma$  is extracted from  $S_e^+$ .

The sets  $S_e$  are uniformly d.c.e.; so  $\mathfrak{A} = \bigcap_e S_e$  is a  $\Pi_2^0(\mathbf{0}')$  class.

If  $\sigma$  is placed into  $S_e^-$  at stage  $s$ , then at that stage there is some  $\sigma' \subset \sigma$  that is marked for  $W_e$ ; we have  $\sigma = \sigma' 0^k 1$  for some  $k < s$ . If  $\sigma$  is placed into  $S_e^+$  at stage

$s$ , then at that stage there is some  $\sigma' \subset \sigma$  that has just lost its mark for  $W_e$ ; in this case,  $\sigma = \sigma'0^s$ . In either case we call  $\sigma'$  the *parent* of  $\sigma$ .

If  $\sigma$  is a string that at stage  $s$  is marked for some  $W_e$ , then at  $s$  we define  $\Gamma(\sigma) = \nu_\sigma 1$  for some  $\nu_\sigma \in 2^{<\omega}$ . Note that this is the only time at which we enumerate  $(\sigma, \tau)$  into  $\Gamma$  (for any  $\tau$ ); so  $\Gamma(\sigma)$  does not change after stage  $s$ .

**Lemma 3.1.** *For  $s < \omega$ , let  $R[s]$  be the collection of all  $\sigma$  that at the end of stage  $s$  are marked active for some  $W_e$  or are in some  $S_e$ . If  $\sigma, \sigma' \in R[s]$  are incomparable, then  $\nu_\sigma$  and  $\nu_{\sigma'}$  are incomparable.*

*Proof.* By induction on  $s$ . Suppose this holds for  $s - 1$ . At stage  $s$ , say we first act for some interval  $\sigma$  which was marked for  $W_e$ . All proper extensions of  $\sigma$  are removed from  $R$ . If  $\sigma', \sigma''$  are two extensions of  $\sigma$  that are now put into  $R$ , then we define  $\nu_\sigma$  and  $\nu_{\sigma'}$  to be incomparable. If  $\sigma' \supset \sigma$  is now put into  $R$  and  $\sigma'' \in R$  is “old”, and  $\sigma' \perp \sigma''$ , then  $\sigma'' \perp \sigma$  and so  $\nu_{\sigma''} \perp \nu_\sigma \subset \nu_{\sigma'}$ .

At the second step of the stage, more intervals are added to  $R$ , but always refining intervals that are already in  $R$ ; if we mark  $\sigma_k = [\sigma 0^k 1]$  for some  $W_{e+1}$  then we define  $\nu_{\sigma_k} = \Gamma(\sigma)\sigma^k$  and so incomparability is assured.  $\square$

Here are some more observations.

**Lemma 3.2.** *For  $s < \omega$ , let  $Q[s]$  be the downward closure of  $\bigcup_{t \leq s} R[t]$ . Then:*

- (1) *Every  $\sigma \in Q[s]$  has some extension in  $R[s]$ .*
- (2)  $2^{<s} \subseteq Q[s]$ .

*Proof.* For (1), note that if  $\sigma$  is removed from  $R$  at stage  $t$  then some extension is put into  $R$  immediately. For (2), note in fact that if  $\sigma$  is removed from  $R$  at stage  $t$  then all extensions of  $\sigma$  of length  $t$  are placed into  $Q$ ; and if  $\sigma \in 2^{<s}$  is placed into  $R$  at stage  $t$  then both extensions of length  $|\sigma| + 1$  are placed into  $R$  at the same stage.  $\square$

**Corollary 3.3.** *Suppose that  $\sigma \in \bigcup_e S_e$  (at the end of time). Then for no  $\sigma'$  do we have  $\Gamma(\sigma') \supseteq \nu_\sigma 0$ .*

*Proof.* Suppose for a contradiction that there is some  $\sigma'$  such that  $\Gamma(\sigma') \supseteq \nu_\sigma 0$ . We cannot have  $\sigma$  and  $\sigma'$  comparable; if  $\sigma' \supset \sigma$  then  $\Gamma(\sigma') \supset \Gamma(\sigma) = \nu_\sigma 1$ ; and if  $\sigma' \subset \sigma$  then  $\Gamma(\sigma') \subseteq \Gamma(\sigma)$ .

Without loss of generality assume that  $\sigma'$  is marked active at stage  $s_0$ ; and suppose that  $\sigma$  is marked active at stage  $s_1$ . We cannot have  $s_1 \leq s_0$ , because then  $\sigma, \sigma' \in R[s_0]$  and so  $\nu_\sigma \perp \nu_{\sigma'}$ . But we also cannot have  $s_0 \leq s_1$ , because then, even if  $\sigma' \notin R[s_1]$ , there is some  $\sigma'' \supseteq \sigma'$  which is in  $R[s_1]$ , and  $\Gamma(\sigma'') \supset \nu_\sigma 0$ , contradicting  $\nu_\sigma \perp \nu_{\sigma''}$ .  $\square$

We now are ready to verify properties (1) and (2). For simplicity of notation, we let (at every stage)  $S_{-1} = \{\langle \rangle\}$ .

**Lemma 3.4.** *Each  $S_e$  is dense, and so  $\mathfrak{A} = \bigcap_e S_e$  is dense too.*

*Proof.* We take some  $\sigma \in S_{e-1}$  and show that  $S_e$  is dense in  $[\sigma]$ . Suppose that  $\sigma$  is put into  $S_{e-1}$  at stage  $s_0$ .

We first note that if some  $[\sigma'] \subset [\sigma]$  is ever marked active for  $W_e$  at a stage  $s \geq s_0$ , then there is a subinterval of  $[\sigma']$  that is (permanently) in  $S_e$ . This is because either no action is taken for  $\sigma'$ , in which case every  $\sigma'0^k 1$  is in  $S_e$ , or action is taken for

$\sigma'$  at some stage  $s$ , in which case  $\sigma'0^s$  is in  $S_e$ . The point is that these markings cannot be eliminated by action below  $\sigma'$ , because such action would remove  $\sigma$  from  $S_{e-1}$  (note that there are no marked strings between  $\sigma$  and  $\sigma'$ ).

Let  $\rho \supset \sigma$  and suppose for a contradiction that  $[\rho] \cap \mathcal{S}_e = \emptyset$ . Let  $s_1 > s_0, |\rho|$ , and find some  $\sigma' \in S_e[s_1]$  compatible with  $\rho$  (so  $\sigma' \supset \sigma$ ). There is a later stage  $s_2$  at which  $\sigma'$  is extracted from  $S_e$ ; at that stage, all successors of  $\sigma'$  at some level  $m > |\rho|$  are marked active for  $W_e$ . At least one of these successors  $\sigma''$  is compatible with  $\rho$ , and so must actually extend  $\rho$ . The string  $\sigma''$  has some extension that is in  $S_e$ , a contradiction.  $\square$

**Lemma 3.5.** *Suppose that  $X \in \mathcal{S}_e$ . Then some initial segment of  $\Gamma(X)$  determines  $W_e$ .*

(Note that we do not assume that  $\Gamma(X)$  is total.)

*Proof.* Suppose that  $\sigma \in S_e$  and  $\sigma \subset X$ . If  $\sigma \in S_e^+$ , then  $\Gamma(\sigma)$  extends some string in  $W_e$ . If  $\sigma \in S_e^-$ , then no  $\tau \supset \Gamma(\sigma')$  is ever enumerated into  $W_e$  (where  $\sigma'$  is  $\sigma$ 's parent; note that  $\sigma'$  is always marked for  $W_e$ ).  $\square$

**Corollary 3.6.**  *$\mathfrak{A} \subseteq \text{dom } \Gamma$  and every  $Y \in \Gamma[\mathfrak{A}]$  is 1-generic.*

*Proof.* The second part follows immediately from the lemma. For the first part, apply the lemma to the sets  $W_{e_n} = 2^{\geq n}$  for  $n \in \omega$ .  $\square$

**Lemma 3.7.**  *$\Gamma[2^\omega]$  is contained in a nowhere dense  $\Pi_1^0(\mathbf{0}')$  class.*

*Proof.* Let  $T$  be the downward closure of the range of  $\Gamma$ , viewed as a relation on strings, i.e.

$$T = \{\tau : \exists \sigma [\tau' \subseteq \Gamma(\sigma)]\}.$$

$T$  is c.e., and so  $[T]$ , the class of paths through  $T$ , is a  $\Pi_1^0(\mathbf{0}')$  class that contains the image of  $\Gamma$  on  $2^\omega$ .

The class  $[T]$  is closed, so to show that it is nowhere dense, it suffices to show that it does not contain any interval. Suppose for a contradiction that  $[\rho]$  is an interval contained in  $[T]$ , which means that every extension of  $\rho$  is in  $T$ . By the definition of  $T$ , there are some  $\sigma$  and  $\tau$  such that  $(\sigma, \tau) \in \Gamma$  and  $\tau \supseteq \rho$ . There is some  $\sigma'$  extending  $\sigma$  which is in  $\bigcup_e S_e$ . Then  $\nu_{\sigma'} \supseteq \tau$ , and  $\nu_{\sigma'}0$  is not on  $T$ ; this is a contradiction.  $\square$

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