Computable Trees, Prime Models, and Relative Decidability

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Abstract

We show that for every computable tree $\mathcal{T}$ with no dead ends and all paths computable, and every $D >_{\mathrm{T}} \emptyset$, there is a D-computable listing of the isolated paths of $\mathcal{T}$. It follows that for every complete decidable theory T such that all the types of T are computable and every $D >_{\mathrm{T}} \emptyset$, there is a D-decidable prime model of T. This result extends a theorem of Csima and yields a stronger version of the theorem, due independently to Slaman and Wehner, that there is a structure with presentations of every nonzero degree but no computable presentation.

1 Introduction

There have been several recent examples of constructions that cannot be performed computably, but can be performed D-computably for any noncomputable D (or every noncomputable D in a particular class, such as the Δ^0_2 sets). A well-known instance of this phenomenon is due independently to Slaman [8] and Wehner [9]. A *presentation* of a countable structure $\mathcal{M}$ is a structure $\mathcal{A} \cong \mathcal{M}$ with universe ω. The *degree* of $\mathcal{A}$ is the Turing degree of the atomic diagram of $\mathcal{A}$.

1.1 Theorem (Slaman; Wehner). *There is a structure with presentations of every nonzero degree but no computable presentation.*

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In Corollary 2.5 below, we show that a strong version of this result can be obtained by considering prime models of complete decidable theories all of whose types are computable. Thus this paper fits into an ongoing program of fine analysis of the computable content of Vaughtian model theory (that is, the study of special models such as prime, homogeneous, and saturated models), which includes work by Csima [1]; Csima, Hirschfeldt, Knight, and Soare [3]; and Csima, Harizanov, Hirschfeldt, and Soare [2]. One of the themes of this program is that the model-theoretic properties of a special model $\mathcal{M}$ can often be used to build a copy $\mathcal{A}$ of $\mathcal{M}$ with some desired computability-theoretic property without explicitly defining an isomorphism between $\mathcal{M}$ and $\mathcal{A}$. This theme is reflected in Theorem 2.1 below and its corollaries.

Another example of a "barely noneffective" construction can be obtained from the following results of Csima [1] and of Goncharov and Nurtazin [4] and independently Millar [7]. For a set D, a structure $\mathcal{A}$ is *D-decidable* if the elementary diagram of $\mathcal{A}$ is D-computable.

1.2 Theorem (Csima). *Let T be a complete decidable theory such that all the types of T are computable and let D be such that $\emptyset <_T D \leqslant_T \emptyset'$. Then T has a D-decidable prime model.*

1.3 Theorem (Goncharov and Nurtazin; Millar). *There is a complete decidable theory T such that all the types of T are computable but T has no computable prime model.*

1.4 Corollary. *There is a complete decidable theory T such that all the types of T are computable and*

1. *T has no computable prime model but*
2. *T has a D-decidable prime model for every D such that $\emptyset <_T D \leqslant_T \emptyset'$.*

It is natural to ask whether Theorem 1.2 can be extended to all noncomputable D. We give a positive answer to this question in Corollary 2.3 below.

In [1], Theorem 1.2 was obtained as a corollary to the following omitting types theorem.

1.5 Theorem (Csima). *Let T be a complete decidable theory, let S be a uniformly computable set of partial types of T, and let D be such that $\emptyset <_T D \leqslant_T \emptyset'$. Then T has a D-decidable model that omits all nonprincipal types in S.*

The extension of this result to non-Δ^0_2 degrees is quite different from that of Theorem 1.2. Theorem 1.5 can be extended to all D of hyperimmune degree, but there is a complete decidable theory T and a uniformly computable set S of partial types of T such that any model of T that omits all nonprincipal types in S has hyperimmune degree. These results and their reverse-mathematical consequences will appear in an upcoming paper by Csima, Hirschfeldt, and Shore.

There is a tight connection between complete decidable theories and computable trees with no dead ends. On the one hand, if T is a complete decidable theory, then for each n the tree $S_n(T)$ of n-types of T is a computable tree with no dead ends, and these trees can be glued together to form a single tree $\mathcal{S}(T)$. On the other hand, any computable tree with no dead ends can be coded into a complete decidable theory. For more details, see Harizanov [5].

The following definition captures those trees that correspond to theories whose types are all computable. (Here $[\mathcal{T}]$ is the set of all paths of the tree $\mathcal{T}$.)

1.6 Definition. A tree $\mathcal{T} \subset 2^{<\omega}$ is a *PAC tree* if it is a computable tree with no dead ends and every path in $[\mathcal{T}]$ is computable. (*PAC* stands for *paths all computable.*)

There is also a tight connection between the effectiveness of prime models of a complete decidable atomic theory T and the effectiveness of listings of the principal types of T, and hence with the effectiveness of listings of the isolated paths of $\mathcal{S}(T)$.

1.7 Definition. Let $S \subset 2^\omega$ and let D be a set. A *D-computable listing* of S is a uniformly D-computable sequence $f_0, f_1, \ldots \in 2^\omega$ such that $S = \{f_n : n \in \omega\}$. (Note that it might be the case that $f_n = f_m$ for some $n \neq m$.)

If S is a set of types of some theory T, then the concept of a D-computable listing of S can be defined analogously.

1.8 Theorem (Goncharov and Nurtazin [4]; Harrington [6]). *Let T be a complete decidable theory and let D be a set. Then T has a D-decidable prime model if and only if there is a D-computable listing of the principal types of T, or equivalently, if and only if there is a D-computable listing of the isolated paths in $[\mathcal{S}(T)]$.*

Theorem 1.3 can be restated in terms of PAC trees as follows.

1.9 Theorem (Goncharov and Nurtazin; Millar). *There is a PAC tree $\mathcal{T}$ such that there is no computable listing of the isolated paths in $[\mathcal{T}]$.*

2 New Results

The following theorem has a simple proof, but as we will see, it can be combined with some of the results mentioned in the previous section to yield several interesting corollaries, including an extension of Theorem 1.2 and a new, simple proof of a strong form of Theorem 1.1.

2.1 Theorem. *Let $\mathcal{T}$ be a PAC tree and let $D >_T \emptyset$. There is a D-computable listing of the isolated paths in $[\mathcal{T}]$.*

Proof. Let $\sigma_0, \sigma_1, \dots$ be a computable enumeration of the elements of $\mathcal{T}$. We build a D-computable listing $f_0, f_1, \dots$ of the isolated paths in $[\mathcal{T}]$ so that $\sigma_n \subset f_n$ for all n. Note that if g is an isolated path in $[\mathcal{T}]$, then there is some n such that g is the unique extension of σ_n in $[\mathcal{T}]$, so if $\sigma_n \subset f_n$ and f_n is a path in $[\mathcal{T}]$, then $f_n = g$. So by ensuring that $\sigma_n \subset f_n \in [\mathcal{T}]$ for all n, we ensure that every isolated path in $[\mathcal{T}]$ is on our listing. Thus our only problem is to guarantee that each f_n is isolated.

The intuitive idea is to use D to pick the path f_n. We extend σ_n until we find a split on $\mathcal{T}$ (if ever). We continue along the right node of this split if $0 \in D$, and along the left node if $0 \notin D$. We then proceed along $\mathcal{T}$ until we find another split (if ever). We continue along the right node of this split if $1 \in D$, and along the left node if $1 \notin D$. We continue defining f_n along $\mathcal{T}$ in this manner, using D to choose which direction to take every time we hit a fork in the road. This ensures that f_n is D-computable (and indeed, that all f_n are uniformly D-computable). But it also ensures that if we hit infinitely many splits along the way while defining f_n, then D can be computed from f_n, which is impossible since f_n is in $[\mathcal{T}]$, and hence is computable, and D is not computable. Thus there are only finitely many splits along f_n, and hence f_n is an isolated path.

The formal definition of f_n is by recursion. Let $m_n^0 = 0$ and $\tau_n^0 = \sigma_n$. (The counter m_n^i keeps track of how many splits have been encountered by step i of the definition.) If τ_n^i has only one immediate successor μ on $\mathcal{T}$, then let $\tau_n^{i+1} = \mu$ and $m_n^{i+1} = m_n^i$. Otherwise, let $\tau_n^{i+1} = \tau_n^i {}^\frown D(m_n^i)$ and $m_n^{i+1} = m_n^i + 1$.

Let $f_n = \bigcup_i \tau_n^i$. It is clear that the f_n are uniformly D-computable, and each $f_n \in [\mathcal{T}]$. Also, $\sigma_n \subset f_n$ for all n, and hence, as explained above, every isolated path in $\mathcal{T}$ is f_n for some n. So all that is left to show is that each f_n is isolated.

Suppose that f_n is not isolated. Then there are infinitely many i such that τ_n^i has two immediate successors. Let $i_0 < i_1 < \cdots$ be all such i. Note that this list is computable, and $m_n^{i_k} = k$ for all k. Furthermore, for each k, the last element of $\tau_n^{i_k+1}$ is $D(m_n^{i_k}) = D(k)$. Thus $D \leqslant_{\mathrm{T}} f_n$. But T is a PAC tree, so $f_n \leqslant_{\mathrm{T}} \emptyset$, and hence $D \leqslant_{\mathrm{T}} \emptyset$, contradicting the choice of D. □

Applying Theorem 1.9, we obtain the following corollary.

2.2 Corollary. *There is a PAC tree $\mathcal{T}$ such that there is no computable listing of the isolated paths of T, but for every $D >_T \emptyset$, there is a D-computable listing of the isolated paths of T.*

By Theorem 1.8, the following extension of Theorem 1.2 is another consequence of Theorem 2.1.

2.3 Corollary. *Let T be a complete decidable theory such that all the types of T are computable and let $D >_T \emptyset$. Then T has a D-decidable prime model.*

Combining this result with Theorem 1.3, we obtain the following corollaries, which extend Corollary 1.4 and Theorem 1.1, respectively.

2.4 Corollary. *There is a complete decidable theory T such that all the types of T are computable and*

1. *T has no computable prime model but*
2. *T has a D-decidable prime model for every $D >_T \emptyset$.*

2.5 Corollary. *There is a structure $\mathcal{A}$ that has no computable presentation but has a D-decidable presentation for every $D >_T \emptyset$. Furthermore, $\mathcal{A}$ is the prime model of a complete decidable theory T such that all the types of T are computable.*

One way in which the structures built by Slaman and by Wehner in proving Theorem 1.1 differ is that Wehner's structure is elementarily equivalent to a computable structure, while Slaman's is not. The structure in Corollary 2.5 is thus more similar to Wehner's structure than to Slaman's, since it is a model of a complete decidable theory, and hence is elementarily equivalent to a decidable structure.

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