

# SUBSPACES OF COMPUTABLE VECTOR SPACES

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ABSTRACT. We show that the existence of a nontrivial proper subspace of a vector space of dimension greater than one (over an infinite field) is equivalent to  $WKL_0$  over  $RCA_0$ , and that the existence of a finite-dimensional nontrivial proper subspace of such a vector space is equivalent to  $ACA_0$  over  $RCA_0$ .

## 1. INTRODUCTION

This paper is a continuation of [3], which is a paper by three of the authors of the present paper. In [3], the effective content of the theory of ideals in commutative rings was studied; in particular, the following computability-theoretic results were established:

- Theorem 1.1.** (1) *There exists a computable integral domain  $R$  that is not a field such that  $\deg(I) \gg \mathbf{0}$  for all nontrivial proper ideals  $I$  of  $R$ .*
- (2) *There exists a computable integral domain  $R$  that is not a field such that  $\deg(I) = \mathbf{0}'$  for all finitely generated nontrivial proper ideals  $I$  of  $R$ .*

These results immediately gave the following proof-theoretic corollaries:

- Corollary 1.2.** (1) *Over  $RCA_0$ ,  $WKL_0$  is equivalent to the statement “Every (infinite) commutative ring with identity that is not a field has a nontrivial proper ideal.”*

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- (2) Over  $\text{RCA}_0$ ,  $\text{ACA}_0$  is equivalent to the statement “Every (infinite) commutative ring with identity that is not a field has a finitely generated nontrivial proper ideal.”

In the present paper, we complement these results with related results from linear algebra. (We refer to [3] for background, motivation, and definitions.)

We start with the following

- Definition 1.3.** (1) A computable field is a computable subset  $F \subseteq \mathbb{N}$  equipped with two computable binary operations  $+$  and  $\cdot$  on  $F$ , together with two elements  $0, 1 \in F$  such that  $(F, 0, 1, +, \cdot)$  is a field.
- (2) A computable vector space (over a computable field  $F$ ) is a computable subset  $V \subseteq \mathbb{N}$  equipped with two computable operations  $+$  :  $V^2 \rightarrow V$  and  $\cdot$  :  $F \times V \rightarrow V$ , together with an element  $0 \in V$  such that  $(V, 0, +, \cdot)$  is a vector space over  $F$ .

This notion was first studied by Dekker [2], then more systematically by Metakides and Nerode [5] and many others.

As in [3] for nontrivial proper ideals in rings, one motivation in the results below is to understand the complexity of nontrivial proper subspaces of a vector space of dimension greater than one, and the proof-theoretic axioms needed to establish their existence. For example, consider the following elementary characterization of when a vector space has dimension greater than one.

**Proposition 1.4.** A vector space  $V$  has dimension greater than one if and only if it has a nontrivial proper subspace.

As in the case of ideals in [3], we will be able to show that this equivalence is not effective, and to pin down the exact proof-theoretic strength of the statement in two versions, for the existence of a nontrivial proper subspace and of a finite-dimensional nontrivial proper subspace:

- Theorem 1.5.** (1) There exists a computable vector space  $V$  of dimension greater than one (over an infinite computable field) such that  $\text{deg}(W) \gg \mathbf{0}$  for all nontrivial proper subspaces  $W$  of  $V$ .
- (2) There exists a computable vector space  $V$  of dimension greater than one (over an infinite computable field) such that  $\text{deg}(W) \geq \mathbf{0}'$  for all finite-dimensional nontrivial proper subspaces  $W$  of  $V$ .

Again, after a brief analysis of the induction needed to establish Theorem 1.5, we obtain the following proof-theoretic corollaries:

- Corollary 1.6.** (1) Over  $\text{RCA}_0$ ,  $\text{WKL}_0$  is equivalent to the statement “Every vector space of dimension greater than one (over an infinite field) has a nontrivial proper subspace.”
- (2) Over  $\text{RCA}_0$ ,  $\text{ACA}_0$  is equivalent to the statement “Every vector space of dimension greater than one (over an infinite field) has a finite-dimensional nontrivial proper subspace.”

## 2. THE PROOF OF THEOREM 1.5

For the proof of part (1) of Theorem 1.5, we begin with a few easy lemmas:

**Lemma 2.1.** *Suppose that  $V$  is a vector space, that  $\{v, w\}$  is a linearly independent set of vectors in  $V$ , and that  $u \neq 0$  is a vector in  $V$ . Then there exists at most one scalar  $\lambda$  such that  $u \in \langle v - \lambda w \rangle$ .*

*Proof.* Suppose that  $u \in \langle v - \lambda_1 w \rangle$  and that  $u \in \langle v - \lambda_2 w \rangle$ . Fix  $\mu_1, \mu_2$  such that  $u = \mu_1(v - \lambda_1 w)$  and  $u = \mu_2(v - \lambda_2 w)$ . Notice that  $\mu_1, \mu_2 \neq 0$  because  $u \neq 0$ . We now have

$$\mu_1 v - \mu_1 \lambda_1 w = u = \mu_2 v - \mu_2 \lambda_2 w,$$

and hence

$$(\mu_1 - \mu_2)v + (\mu_2 \lambda_2 - \mu_1 \lambda_1)w = 0.$$

Since  $\{v, w\}$  is linearly independent, it follows that  $\mu_1 - \mu_2 = 0$  and  $\mu_2 \lambda_2 - \mu_1 \lambda_1 = 0$ , hence  $\mu_1 = \mu_2$  and  $\mu_1 \lambda_1 = \mu_2 \lambda_2$ . Since  $\mu_1 = \mu_2 \neq 0$ , it follows from the second equation that  $\lambda_1 = \lambda_2$ .  $\square$

**Lemma 2.2.** *Suppose that  $V$  is a vector space with basis  $B$ , which is linearly ordered by  $\prec$ . Suppose that*

- (1)  $v \in V$ .
- (2)  $e \in B$ .
- (3)  $\lambda$  is a scalar.
- (4)  $e \succ \max(\text{supp}(v))$  (where  $\text{supp}(v) = \text{supp}_B(v)$ , the support of  $v$ , is the finite set of basis vectors in  $B$  needed to write  $v$  as a linear combination in this basis).

*Then  $B \setminus \{e\}$  is a basis for  $V$  over  $\langle e - \lambda v \rangle$ , and, for all  $w \in V$ ,  $\max(\text{supp}_{B \setminus \{e\}}(w + \langle e - \lambda v \rangle)) \preceq \max(\text{supp}_B(w))$ .*

*Proof.* Notice that  $e \in \langle (B \setminus \{e\}) \cup \{e - \lambda v\} \rangle$  because  $e \notin \text{supp}(v)$ , so  $(B \setminus \{e\}) \cup \{e - \lambda v\}$  spans  $V$ . Suppose that  $e_1, e_2, \dots, e_n \in B \setminus \{e\}$  are distinct and  $\mu_1, \mu_2, \dots, \mu_n$  are scalars such that

$$\mu_1 e_1 + \mu_2 e_2 + \dots + \mu_n e_n \in \langle e - \lambda v \rangle.$$

Fix  $\mu$  such that

$$\mu_1 e_1 + \mu_2 e_2 + \cdots + \mu_n e_n = \mu(e - \lambda v)$$

and notice that we must have  $\mu = 0$  (by looking at the coefficient of  $e$ ), hence each  $\mu_i = 0$  because  $B$  is a basis. Therefore,  $B \setminus \{e\}$  is a basis for  $V$  over  $\langle e - \lambda v \rangle$ . By hypothesis 4, the last line of the lemma now follows easily.  $\square$

**Lemma 2.3.** *Suppose that  $V$  is a vector space with basis  $B$ , which is linearly ordered by  $\prec$ . Suppose that*

- (1)  $v_1, v_2 \in V$ .
- (2)  $e_1, e_2 \in B$  with  $e_1 \neq e_2$ .
- (3)  $\lambda$  is a scalar.
- (4)  $e_1 \succ \max(\text{supp}(v_1) \cup \text{supp}(v_2))$ .
- (5)  $\{v_1, e_1\}$  is linearly independent.
- (6)  $v_1 \notin \langle e_2 - \lambda v_2 \rangle$ .

Then  $\{v_1, e_1\}$  is linearly independent over  $\langle e_2 - \lambda v_2 \rangle$ .

*Proof.* Suppose that

$$\mu_1 v_1 + \mu_2 e_1 = \mu_3(e_2 - \lambda v_2).$$

We need to show that  $\mu_1 = \mu_2 = 0$ .

*Case 1:*  $e_1 \prec e_2$ . In this case, we must have  $\mu_3 = 0$  (by looking at the coefficient of  $e_2$ ). Thus,  $\mu_1 v_1 + \mu_2 e_1 = 0$ , and hence  $\mu_1 = \mu_2 = 0$  since  $\{v_1, e_1\}$  is linearly independent.

*Case 2:*  $e_1 \succ e_2$ . In this case, we must have  $\mu_2 = 0$  (by looking at the coefficient of  $e_1$ ). Thus,  $\mu_1 v_1 = \mu_3(e_2 - \lambda v_2)$ . Since  $v_1 \notin \langle e_2 - \lambda v_2 \rangle$ , this implies that  $\mu_1 = 0$ .  $\square$

By applying the above three lemmas in the corresponding quotient, we obtain the following results.

**Lemma 2.4.** *Suppose that  $V$  is a vector space, that  $X \subseteq V$ , that  $\{v, w\}$  is linearly independent over  $\langle X \rangle$ , and that  $u \notin \langle X \rangle$ . Then there exists at most one  $\lambda$  such that  $u \in \langle X \cup \{v - \lambda w\} \rangle$ .  $\square$*

**Lemma 2.5.** *Suppose that  $V$  is a vector space, that  $X \subseteq V$ , and that  $B$  is a basis for  $V$  over  $\langle X \rangle$  that is linearly ordered by  $\prec$ . Suppose that*

- (1)  $v \in V$ .
- (2)  $e \in B$ .
- (3)  $\lambda$  is a scalar.
- (4)  $e \succ \max(\text{supp}(v))$ .

Then  $B \setminus \{e\}$  is a basis for  $V$  over  $\langle X \cup \{e - \lambda v\} \rangle$  and, for all  $w \in V$ ,  $\max(\text{supp}_{B \setminus \{e\}}(w + \langle X \cup \{e - \lambda v\} \rangle)) \preceq \max(\text{supp}_B(w))$ .

Then  $B \setminus \{e\}$  is a basis for  $V$  over  $\langle e - \lambda v \rangle$ , and, for all  $w \in V$ ,  $\max(\text{supp}_{B \setminus \{e\}}(w + \langle e - \lambda v \rangle)) \preceq \max(\text{supp}_B(w))$ .  $\square$

**Lemma 2.6.** *Suppose that  $V$  is a vector space, that  $X \subseteq V$ , and that  $B$  is a basis for  $V$  over  $\langle X \rangle$  that is linearly ordered by  $\prec$ . Suppose that*

- (1)  $v_1, v_2 \in V$ .
- (2)  $e_1, e_2 \in B$  with  $e_1 \neq e_2$ .
- (3)  $\lambda$  is a scalar.
- (4)  $e_1 \succ \max(\text{supp}(v_1) \cup \text{supp}(v_2))$ .
- (5)  $\{v_1, e_1\}$  is linearly independent over  $\langle X \rangle$ .
- (6)  $v_1 \notin \langle X \cup \{e_2 - \lambda v_2\} \rangle$ .

Then  $\{v_1, e_1\}$  is linearly independent over  $\langle X \cup \{e_2 - \lambda v_2\} \rangle$ .  $\square$

*Proof of Theorem 1.5.* Fix two disjoint c.e. sets  $A$  and  $B$  such that  $\deg(S) \gg \mathbf{0}$  for any set  $S$  satisfying  $A \subseteq S$  and  $B \cap S = \emptyset$ . Let  $V^\infty$  be the vector space over the infinite computable field  $F$  on the basis  $e_0, e_1, e_2, \dots$  (ordered by  $\prec$  as listed) and list  $V^\infty$  as  $v_0, v_1, v_2, \dots$  (viewed as being coded effectively by natural numbers). We may assume that  $v_0$  is the zero vector of  $V^\infty$ . Fix a computable injective function  $g: \mathbb{N}^3 \rightarrow \mathbb{N}$  such that  $e_{g(i,j,n)} \succ \max(\text{supp}(v_i) \cup \text{supp}(v_j))$  for all  $i, j, n \in \mathbb{N}$ . We build a computable subspace  $U$  of  $V^\infty$  with the plan of taking the quotient  $V = V^\infty/U$ .

We have the following requirements for all  $v_i, v_j \notin U$ :

- $$R_{i,j,n} : n \notin A \cup B \Rightarrow \text{each of } \{v_i, e_{g(i,j,n)}\} \text{ and } \{v_j, e_{g(i,j,n)}\}$$
- $$\text{are linearly independent over } U,$$
- $$n \in A \Rightarrow e_{g(i,j,n)} - \lambda v_i \in U \text{ for some nonzero } \lambda \in F, \text{ and}$$
- $$n \in B \Rightarrow e_{g(i,j,n)} - \lambda v_j \in U \text{ for some nonzero } \lambda \in F.$$

We now effectively build a sequence  $U_2, U_3, U_4, \dots$  of finite subsets of  $V$  such that  $U_2 \subseteq U_3 \subseteq U_4 \subseteq \dots$ , and we set  $U = \bigcup_{n \geq 2} U_n$ . We also define a function  $h: \mathbb{N}^4 \rightarrow \{0, 1\}$  for which  $h(i, j, n, s) = 1$  if and only if we have acted for requirement  $R_{i,j,n}$  at some stage  $\leq s$  (as defined below). We ensure that for all  $k \geq 2$ , we have  $v_k \in U$  if and only if  $v_k \in U_k$ , which will make our set  $U$  computable. We begin by letting  $U_2 = \{v_0\}$  and letting  $h(i, j, n, s) = 0$  for all  $i, j, n, s$  with  $s \leq 2$ . Suppose that  $s \geq 2$  and we have defined  $U_s$  and  $h(i, j, n, s)$  for all  $i, j, n$ . Suppose also that we have for any  $i, j, n$ , and  $s$  such that  $v_i, v_j \notin \langle U_s \rangle$ :

- (1) If  $h(i, j, n, s) = 0$ , then each of  $\{v_i, e_{g(i,j,n)}\}$  and  $\{v_j, e_{g(i,j,n)}\}$  is linearly independent over  $\langle U_s \rangle$ .
- (2) If  $h(i, j, n, s) = 1$  and  $n \in A_s$ , then  $e_{g(i,j,n)} - \lambda v_i \in U_s$  for some nonzero  $\lambda \in F$ .

- (3) If  $h(i, j, n, s) = 1$  and  $n \in B_s$ , then  $e_{g(i,j,n)} - \lambda v_j \in U_s$  for some nonzero  $\lambda \in F$ .

Check whether there exists a triple  $\langle i, j, n \rangle < s$  (under some effective coding) such that

- (1)  $v_i, v_j \notin \langle U_s \rangle$ .
- (2)  $n \in A_s \cup B_s$ .
- (3)  $h(i, j, n, s) = 0$ .

Suppose first that no such triple  $\langle i, j, n \rangle$  exists. If  $v_{s+1} \in \langle U_s \rangle$ , then let  $U_{s+1} = U_s \cup \{v_{s+1}\}$ , otherwise let  $U_{s+1} = U_s$ . Also, let  $h(i, j, n, s+1) = h(i, j, n, s)$  for all  $i, j, n$ .

Suppose then that such a triple  $\langle i, j, n \rangle$  exists, and fix the least such triple. If  $n \in A_s$ , then search for the least (under some effective coding) nonzero  $\lambda \in F$  such that  $v_k \notin \langle U_s \cup \{e_{g(i,j,n)} - \lambda v_i\} \rangle$  for all  $k \leq s$  such that  $v_k \notin U_s$ . (Such  $\lambda$  must exist by Lemma 2.4 and the fact that  $F$  is infinite.) Let  $U'_s = U_s \cup \{e_{g(i,j,n)} - \lambda v_i\}$  and let  $h(i, j, n, s+1) = 1$ . If  $n \in B_s$ , then proceed likewise with  $v_j$  replacing  $v_i$ . Now, if  $v_{s+1} \in \langle U'_s \rangle$ , then let  $U_{s+1} = U'_s \cup \{v_{s+1}\}$ ; otherwise let  $U_{s+1} = U'_s$ . Also, let  $h(i, j, n, s+1) = h(i, j, n, s)$  for all other  $i, j, n$ . Using Lemma 2.6, it follows that our inductive hypothesis is maintained, so we may continue.

We can now view the quotient space  $V = V^\infty/U$  as the set of  $<_{\mathbb{N}}$ -least representatives (which is a computable subset of  $V^\infty$ ). Notice that  $V$  is not one-dimensional because  $\{v_1, e_{g(1,2,n)}\}$  is linearly independent over  $U$  for any  $n \notin A \cup B$  (since  $v_1, v_2 \notin U$ ). Suppose that  $W$  is a nontrivial proper subspace of  $V$ , and fix  $W_0$  such that  $W = W_0/U$ . Then  $W_0$  is a  $W$ -computable subspace of  $V^\infty$ , and  $U \subset W_0 \subset V^\infty$ . Fix  $v_i, v_j \in V^\infty \setminus U$  such that  $v_i \in W_0$  and  $v_j \notin W_0$ . Let  $S = \{n : e_{g(i,j,n)} \in W_0\}$ . We then have that  $S \leq_T W_0 \equiv_T W$ , that  $A \subseteq S$ , and that  $B \cap S = \emptyset$ . Thus  $\deg(S) \gg \mathbf{0}$ , establishing part (1) of Theorem 1.5.

Part (2) of Theorem 1.5 now follows easily from part (1) and Arslanov's Completeness Criterion [1]: If  $W$  is a finite-dimensional nontrivial proper subspace of the above vector space  $V$  then  $W_0$  is a c.e. set that computes a degree  $\gg \mathbf{0}$ ; thus  $\deg(W)$  must equal  $\mathbf{0}'$ .  $\square$

### 3. THE PROOF OF COROLLARY 1.6

As usual for these arguments, we only have to check that

- (i)  $\text{WKL}_0$  (or  $\text{ACA}_0$ , respectively) suffices to prove the existence of a (finite-dimensional) nontrivial proper subspace (establishing the left-to-right direction of Corollary 1.6); and
- (ii) the above computability-theoretic arguments can be carried out in  $\text{RCA}_0$  (establishing the right-to-left direction of Corollary 1.6).

Part (i) just requires a bit of coding. Using  $\text{WKL}_0$ , one can code membership in a nontrivial proper subspace  $W$  of a vector space  $V$  on a binary tree  $T$  where one arbitrarily fixes two linearly independent vectors  $w, w' \in V$  such that  $w \in W$  and  $w' \notin W$  is specified. A node  $\sigma \in T_W$  is now terminal if the subspace axioms for  $W$  are violated along  $\sigma$  using coefficients with Gödel number  $< |\sigma|$ , which can be checked effectively relative to the open diagram of the vector space. Using  $\text{ACA}_0$ , one can form the one-dimensional subspace generated by any nonzero vector in  $V$ .

Part (ii) boils down to checking that  $\Sigma_1^0$ -induction suffices for the computability-theoretic arguments from Section 2. First of all, note that the definition of  $U$  and of the vector space operations on  $U$  can be carried out using  $\Delta_1^0$ -induction.  $\text{WKL}_0$  is equivalent to showing  $\Sigma_1^0$ -Separation, so fix any sets  $A$  and  $B$  that are  $\Sigma_1^0$ -definable in our model of arithmetic. Then their enumerations  $\{A_s\}_{s \in \omega}$  and  $\{B_s\}_{s \in \omega}$  exist in the model, and from them we can define the subspace  $U$ , the quotient space  $V = V^\infty/U$ , and the function mapping each vector  $v \in V^\infty$  to its  $<_{\mathbb{N}}$ -least representative modulo  $U$ , using only  $\Sigma_1^0$ -induction. (The latter function only requires that in  $\text{RCA}_0$ , any infinite  $\Delta_1^0$ -definable set can be enumerated in order.) The hypothesis now provides the nontrivial proper subspace  $W$ , and from it we can define the separating set  $S$  by  $\Delta_1^0$ -induction.

Proving the right-to-left direction of Corollary 1.6 (2) could be done using the concept of maximal pairs of c.e. sets as in our companion paper [3]. But for vector spaces, there is actually a much simpler proof: In the above construction, simply set  $A$  to be any  $\Sigma_1^0$ -set and  $B = \emptyset$ . Now  $V$  must be a vector space of dimension greater than one. Since any finitely generated nontrivial proper subspace can compute a one-dimensional subspace, we may assume we are given a one-dimensional subspace  $W$ , spanned by  $v_i$ , say. But then

$$\begin{aligned} n \in A &\text{ iff } \{v_i, e_{g(i,1,n)}\} \text{ is linearly dependent in } V \\ &\text{ iff } e_{g(i,1,n)} \in W, \end{aligned}$$

and so  $W$  can compute  $A$  as desired.

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