1 (The Ackermann Model). Let the binary relation $\in$ on the natural numbers $\omega$ be defined by $a \in b$ iff $a$ is an exponent in the (unique) expansion of $b$ as a sum of distinct powers of 2. (For example, $11 = 2^0 + 2^1 + 2^3$, so $0 \in 11$, $1 \in 11$, and $3 \in 11$.)

Let $A = (\omega, \in)$. Which of the axioms of ZFC does $A$ satisfy? (For each axiom of ZFC, prove either that $A$ satisfies it or that it does not.)

2. Let $R(0) = \omega$ and, for each $n \geq 0$, let $R(n + 1) = \mathcal{P}(R(n))$. Let $R = \bigcup_{n \in \omega} R(n)$ and let $Z = (R, \in)$, where $\in$ has its usual meaning.

Which of the axioms of ZFC does $Z$ satisfy? (For each axiom of ZFC, prove either that $Z$ satisfies it or that it does not.)

3. Let $S$ be defined as usual for ordinals: $S(\alpha) = \alpha \cup \{\alpha\}$. Show that ZF proves that there is a set $N$ of ordinals such that:

1. $N$ contains every natural number (i.e., every finite ordinal) and

2. for any property $P$ of ordinals definable in the language of set theory, if $P(\emptyset)$ and $P(\alpha) \rightarrow P(S(\alpha))$, then $P(\alpha)$ for all $\alpha \in N$. 
4. A subset $X$ of a limit ordinal $\gamma$ is *unbounded* if for every $\alpha < \gamma$, there is a $\beta \in X$ such that $\alpha < \beta$.

   a. Show that if $\alpha$ is an ordinal, $\gamma$ is a limit ordinal, and $X \subseteq \gamma$ is unbounded, then $\alpha + \gamma = \sup_{\beta \in X} (\alpha + \beta)$.

   b. Use transfinite induction to show that if $\alpha$ and $\delta$ are ordinals, then there is an ordinal $\beta$ such that $\alpha + \beta = \delta$.

   c. Show that there are ordinals $\beta < \delta$ such that there is no ordinal $\alpha$ with $\alpha + \beta = \delta$.

5. **(take-home exam problem).** For this problem, note that a *function* is a set $f$ of ordered pairs such that for each $x$ there is at most one $y$ with $\langle x, y \rangle \in f$. (As usual, if such a $y$ exists, we denote $y$ by $f(x)$.) The *domain* of $f$ is the set of all $x$ such that $\langle x, y \rangle \in f$ for some $y$.

   Let $I$ be a nonempty set and $f$ a function such that $f(i)$ is nonempty for each $i \in I$. The *Cartesian product* of $\{f(i) : i \in I\}$ is the set of all functions $g$ with domain $I$ such that $g(i) \in f(i)$ for every $i \in I$.

   Show that the following assertions are equivalent over ZF. Be careful to note exactly which axioms of ZF you are using in your proofs.

   1. For every set $A$ of pairwise disjoint nonempty sets there exists a set $C$ such that $C \cap x$ has exactly one element for each $x \in A$.

   2. For every set $A$ of nonempty sets there is a function $f$ with domain $A$ such that $f(x) \in x$ for each $x \in A$.

   3. Let $I$ be a nonempty set and $f$ a function such that $f(i)$ is nonempty for each $i \in I$. Then the Cartesian product of $\{f(i) : i \in I\}$ is nonempty.
The following is not a problem to hand in, just an interesting question to think about, which wasn’t included in the assignment to avoid making it too long:

Add constant symbols $c_0, c_1, \ldots$ to the language of set theory. Let $T$ be ZFC together with axioms that state that each $c_n$ is a finite ordinal, and $c_0 \ni c_1 \ni c_2 \cdots$. Assuming ZFC is consistent, it follows by compactness that $T$ is consistent and hence has a model $\mathcal{M}$. By the axioms of infinity and comprehension there is a $y \in \mathcal{M}$ whose elements are exactly the finite ordinals. Then $y$ is itself an ordinal (indeed, the first infinite ordinal) in $\mathcal{M}$. But $y$ contains elements $z_0 \ni z_1 \ni z_2 \ni \cdots$. Explain why this fact contradicts neither the fact that $\mathcal{M}$ satisfies foundation, nor the fact that $y$ is an ordinal in $\mathcal{M}$. 