Math 27800 / CS 27800, Winter 2024: Assignment 1

Due Friday, January 19th

Problem 5 is a "take-home exam" problem, to be worked out individually, without consultation with other students or anyone else. It is fine to ask me clarifying questions about this problem. For all other problems, collaboration is encouraged, but each person should write their own solutions independently.

1 (The Ackermann Model). Let the binary relation $\bar{\in}$ on the natural numbers ω be defined by $a \,\bar{\in}\, b$ iff a is an exponent in the (unique) expansion of b as a sum of distinct powers of 2. (For example, $11 = 2^0 + 2^1 + 2^3$, so $0 \,\bar{\in}\, 11$, $1 \,\bar{\in}\, 11$, and $3 \,\bar{\in}\, 11$.)

Let $\mathcal{A} = (\omega, \bar{\in})$. Which of the axioms of ZFC does $\mathcal{A}$ satisfy? (For each axiom of ZFC, prove either that $\mathcal{A}$ satisfies it or that it does not.)

2. Let $R(0) = \omega$ and, for each $n \geqslant 0$, let $R(n+1) = \mathcal{P}(R(n))$. Let $R = \bigcup_{n\in\omega} R(n)$ and let $\mathcal{Z} = (R, \in)$, where $\in$ has its usual meaning.

Which of the axioms of ZFC does $\mathcal{Z}$ satisfy? (For each axiom of ZFC, prove either that $\mathcal{Z}$ satisfies it or that it does not.)

3. Recall that an ordinal is a set α that is transitive (i.e., if $\beta \in \alpha$ then $\beta \subset \alpha$) and such that the restriction of $\in$ to the elements of α is a well-order. Let the successor relation S be defined as usual for ordinals: $S(\alpha) = \alpha \cup \{\alpha\}$. Recall that $0 = \emptyset,\ 1 = S(0),\ 2 = S(1),\ \ldots$

Show that ZF proves that there is an ordinal ω such that:

1. ω contains $0, 1, \ldots$ and
2. for any property P of ordinals definable in the language of set theory, if $P(\emptyset)$ and $P(\alpha) \to P(S(\alpha))$, then $P(\alpha)$ for all $\alpha \in \omega$.

Hint: $0, 1, \ldots$ *are finite ordinals.*

4. Add constant symbols $c_0, c_1, \ldots$ to the language of set theory. Let T be ZFC together with axioms that state that each c_n is a finite ordinal, and $c_0 \ni c_1 \ni c_2 \cdots$. Assuming ZFC is consistent, it follows by compactness that T is consistent and hence has a model $\mathfrak{M}$. Since $\mathfrak{M}$ is a model of ZF, it contains an element ω with the properties in Problem 3. Then ω is itself an ordinal (indeed, the first infinite ordinal) in $\mathfrak{M}$. But ω contains elements $z_0 \ni z_1 \ni z_2 \ni \cdots$. Explain why this fact contradicts neither the fact that $\mathfrak{M}$ satisfies foundation, nor the fact that ω is an ordinal in $\mathfrak{M}$.

5 (take-home exam problem). For this problem, note that a *function* is a set f of ordered pairs such that for each x there is at most one y with $\langle x, y\rangle \in f$. (As usual, if such a y exists, we denote y by $f(x)$.) The *domain* of f is the set of all x such that $\langle x, y\rangle \in f$ for some y.

Let I be a nonempty set and f a function such that $f(i)$ is nonempty for each $i \in I$. The *Cartesian product* of $\{f(i) : i \in I\}$ is the set of all functions g with domain I such that $g(i) \in f(i)$ for every $i \in I$.

Show that the following assertions are equivalent over ZF. Be careful to note exactly which axioms of ZF you are using in your proofs.

1. For every set A of pairwise disjoint nonempty sets there exists a set C such that $C \cap x$ has exactly one element for each $x \in A$.

2. For every set A of nonempty sets there is a function f with domain A such that $f(x) \in x$ for each $x \in A$.

3. Let I be a nonempty set and f a function such that $f(i)$ is nonempty for each $i \in I$. Then the Cartesian product of $\{f(i) : i \in I\}$ is nonempty.