Math 27800 / CS 27800, Winter 2024: Assignment 4
Due Friday, February 9th

Problem 4 is a "take-home exam" problem, to be worked out individually.

1. Show that there is no quantifier-free formula $\varphi(x, y)$ in the language of $\mathfrak{N}_S = (\mathbb{N}; 0, S)$ such that $\mathfrak{N}_s \vDash \varphi(m, n)$ iff $m < n$. (In other words, the $<$ relation is not definable in $\mathfrak{N}_s$ by a quantifier-free formula.)

2. Fix a theory T. A *literal* is a formula that is either atomic or is the negation of an atomic formula. Suppose that for every conjunction of finitely many literals $\psi(x, \vec{y})$, there is a quantifier-free formula $\theta(\vec{y})$ such that

$$T \vdash (\exists x \psi(x, \vec{y})) \leftrightarrow \theta(\vec{y}).$$

Show that for *every* formula $\rho(\vec{y})$ there is a quantifier-free formula $\nu(\vec{y})$ such that

$$T \vdash \rho(\vec{y}) \leftrightarrow \nu(\vec{y}).$$

[Proposition 2.6.1 in Avigad might be helpful here. You may assume that proposition in your solution. Here $\vec{y}$ is just a tuple of variables.]

3. Show that the theory of the natural numbers $(\mathbb{N}; 0, S, <, +, \times)$ has $2^{\aleph_0}$ many nonisomorphic countable models. *[Hint: For each set of primes A, construct a model containing an element whose prime divisors are exactly the elements of A.]*

4 (take-home exam problem). Let T be a theory in a language that includes the constant symbol 0, the unary function symbol S, and the ternary relation symbol H. Let $\mathbf{n}$ denote the numeral $S \cdots S0$, where there are n many S's. We say that T is *ω-complete* if for any formula $\varphi(x)$, if $\varphi(\mathbf{n}) \in T$ for every $n \in \mathbb{N}$, then $\forall x \varphi(x) \in T$. Suppose that $T \vdash H(\mathbf{e}, \mathbf{m}, \mathbf{t})$ iff the eth Turing machine on input m runs for exactly t many steps and then halts. Suppose that T is axiomatizable and ω-complete. Show that T is not complete.