Math 27800 / CS 27800, Winter 2024: Assignment 5
Due Friday, February 16th

Problem 2 is a "take-home exam" problem, to be worked out individually.

1. Show that a theory is computably enumerable iff it is axiomatizable. *[Hint: Any formula φ is equivalent to $\varphi \wedge \varphi \wedge \cdots \wedge \varphi$.]*

2 (take-home exam problem). Working in a language with equality, countably many unary relation symbols $R_0, R_1, \ldots$, and no function or constant symbols, let Γ consist of the following axioms:

$$\exists x\,(R_i x \wedge \forall y\,(R_i y \to y = x)) \qquad \text{for each } i \in \omega$$

and

$$\forall x\,(R_i x \to \neg R_j x) \qquad \text{for each } i, j \in \omega \text{ such that } i \neq j.$$

Let $T = \mathrm{Cn}(\Gamma)$. Prove that T admits elimination of quantifiers.

3. Let $\mathcal{A}$ be the directed graph consisting of exactly one n-cycle for each $n > 1$ (with all of the cycles disjoint, and no other vertices or edges than the ones in these cycles). Let T be the theory of $\mathcal{A}$.

a. Show that T does not admit elimination of quantifiers (i.e., find a formula φ for which there is no quantifier-free formula ψ for which $T \vdash \varphi \leftrightarrow \psi$).

b. Show that T is axiomatizable but not finitely axiomatizable.

4. Do Exercise 12.2.1 in Avigad. You may assume Exercise 12.1.1. The definition of ω-consistent is given in Definition 12.2.2.